Adiabatic Atmosphere

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1 Adiabatic ideal gas

For an ideal gas at constant temperature $PV = \text{const.}$ (This is an immediate consequence of the ideal gas law.) Suppose instead that the ideal gas undergoes a change in which there is no heat flow. That is called adiabatic. Then $PV^\gamma = \text{const.}$, with $\gamma = C_P/C_V \approx 7/5$ for air. We are more interested in the density than the volume. Since the volume and density are inversely related, the adiabatic relation becomes $P \propto \rho^\gamma$.

It will turn out to be convenient to fit the temperature into this also. If $PV^\gamma = \text{const.}$, and $PV = nRT$, then $PVV^{\gamma-1} = \text{const.}$, and $TV^{\gamma-1} = \text{const.}$. Or, in terms of the density, $T \propto \rho^{\gamma-1}$.

2 Adiabatic atmosphere

As we will discuss in lecture, the atmosphere in which the pressure and the density are related by $P \propto \rho^\gamma$ is just stable. Let us assume that the pressure and the density are so related at each elevation $z$. The ground is $z = 0$.

Now we combine $-\partial_z P = \rho g$ with $P \propto \rho^\gamma$:

$$P' \propto \rho^{\gamma-1} \rho'$$  \hspace{1cm} (1)

$$-\rho g \propto \rho^{\gamma-1} \rho'$$  \hspace{1cm} (2)

$$g \propto \rho^{\gamma-2} \rho'.$$  \hspace{1cm} (3)

Using $T \propto \rho^{\gamma-1}$, we see that the RHS is proportional to $T'$. Thus, we have the very simple result that $T' = \text{const.}$! That means that $T$ is of the form $a + bz$. It is more convenient to write this in the physically relevant way

$$T(z) = T_0 \left( \frac{z_0 - z}{z_0} \right).$$  \hspace{1cm} (4)

$T_0$ is the temperature at the ground $z = 0$, and $z_0$ is the top of the atmosphere where $T \to 0$. You should draw yourself a graph of this function.
Using the adiabatic relations $T \propto \rho^{\gamma-1}$ and $P \propto \rho^{\gamma}$, we obtain the corresponding results

$$\rho(z) = \rho_0 \left( \frac{z_0 - z}{z_0} \right)^{1/(\gamma-1)}$$  \hspace{1cm} (5)

and

$$P(z) = P_0 \left( \frac{z_0 - z}{z_0} \right)^{\gamma/(\gamma-1)}.$$ \hspace{1cm} (6)

Draw pictures of these two too. Hint: It is important to consider whether the powers on the RHS’s are positive or negative or greater or smaller than 1.

All these functions vanish at $z = z_0$ which is, therefore, the top of the atmosphere.

### 3 Problem

Problem: Use the known values of $P_0$ and $\rho_0$ to determine an expression and a numerical value for $z_0$. Then do the same for the constant $T'$, which is called the adiabatic lapse rate. How much cooler should it be at Lake Tahoe than in Davis?