Adiabatic Atmosphere

Physics 9HB, Joe Kiskis

1 Adiabatic ideal gas

For an ideal gas at constant temperature $PV = const.$ (This is an immediate consequence of the ideal gas law.) Suppose instead that the ideal gas undergoes a change in which there is no heat flow. That is called *adiabatic*. Then $PV^\gamma = const.$, with $\gamma = C_P/C_V \approx 7/5$ for air. We are more interested in the density than the volume. Since the volume and density are inversely related, the adiabatic relation becomes $P \propto \rho^\gamma$.

It will turn out to be convenient to fit the temperature into this also. If $PV^\gamma = const.$, and $PV = nRT$, then $PVV^{\gamma-1} = const.$, and $TV^{\gamma-1} = const.$ Or, in terms of the density, $T \propto \rho^{\gamma-1}$.

2 Adiabatic atmosphere

As we will discuss in lecture, the atmosphere in which the pressure and the density are related by $P \propto \rho^\gamma$ is just stable. Let us assume that the pressure and the density are so related at each elevation z. The ground is $z = 0$.

Now we combine $-\partial_z P = \rho g$ with $P \propto \rho^\gamma$:

$$\begin{aligned} P' &\propto \rho^{\gamma-1}\rho' && (1) \\ -\rho g &\propto \rho^{\gamma-1}\rho' && (2) \\ g &\propto \rho^{\gamma-2}\rho'. && (3) \end{aligned}$$

Using $T \propto \rho^{\gamma-1}$, we see that the RHS is proportional to T'. Thus, we have the very simple result that $T' = const.$!! That means that T is of the form $a + bz$. It is more convenient to write this in the physically relevant way

$$T(z) = T_0\left(\frac{z_0 - z}{z_0}\right). \quad (4)$$

T_0 is the temperature at the ground $z = 0$, and z_0 is the top of the atmosphere where $T \to 0$. You should draw yourself a graph of this function.

Using the adiabatic relations $T \propto \rho^{\gamma-1}$ and $P \propto \rho^{\gamma}$, we obtain the corresponding results

$$\rho(z) = \rho_0(\frac{z_0 - z}{z_0})^{1/(\gamma-1)} \tag{5}$$

and

$$P(z) = P_0(\frac{z_0 - z}{z_0})^{\gamma/(\gamma-1)}. \tag{6}$$

Draw pictures of these two too. Hint: It is important to consider whether the powers on the RHS's are positive or negative or greater or smaller than 1.

All these functions vanish at $z = z_0$ which is, therefore, the top of the atmosphere.

3 Problem

Problem: Use the known values of P_0 and ρ_0 to determine an expression and a numerical value for z_0. Then do the same for the constant T', which is called the *adiabatic lapse rate*. How much cooler should it be at Lake Tahoe than in Davis?