Fluid kinematics

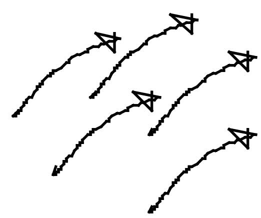
Physical quantities

First we need a way to describe the state of the fluid. The fluid fills up a region of space and time. So quantities that describe it are functions of t and **x.** The values of the physical quantities depend on when and where you look. There is the density ρ, the pressure P, and the velocity **v**. All these are functions of t and **x**. The density and pressure are scalar fields. The velocity is a vector field. It is vector attached to each space-time point. The vector is the velocity at that time and place.

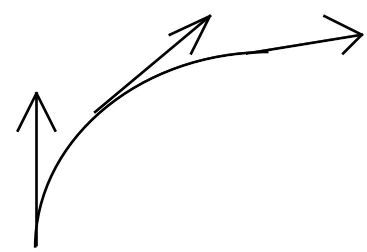
Streamlines

Consider the vector field **v**(t,**x**) at a fixed time. Form curves by joining these vectors. Start at the point $\mathbf{x}_0$, and move to the points $\mathbf{x}_1$ and $\mathbf{x}_2$ with the construction

$$
\begin{aligned}
\mathbf{x}_1 &= \mathbf{x}_0 + \varepsilon \mathbf{v}(\mathbf{x}_0) \\
\mathbf{x}_2 &= \mathbf{x}_1 + \varepsilon \mathbf{v}(\mathbf{x}_1) \\
&\vdots
\end{aligned}
$$



In the limit of small ε, this gives a curve **x**(s) which has as it tangents the velocity. Explicitly, the tangent to **x**(s) is

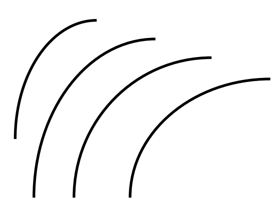


$$
\begin{aligned}
\frac{d\mathbf{x}(s)}{ds} &= \frac{1}{\varepsilon}\left[\mathbf{x}(s+\varepsilon) - \mathbf{x}(s)\right] \\
&= \frac{1}{\varepsilon}\left[\mathbf{x}(s) + \varepsilon \mathbf{v}(\mathbf{x}(s)) - \mathbf{x}(s)\right] \\
&= \mathbf{v}(\mathbf{x}(s))
\end{aligned}
$$

All this is at fixed t. Thus, to construct the *streamline* $\mathbf{x}(\mathbf{x}_0, s)$ through $\mathbf{x}_0$ at time t, one must solve the system of ordinary differential equations

$$\frac{d\mathbf{x}(s)}{ds} = \mathbf{v}(t, \mathbf{x}(s))$$

at fixed t. The collection of all these streamlines makes a picture.



A particle in the fluid follows a path which has a velocity

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(t, \mathbf{x}(t)) \quad .$$

This is similar to the previous equation, but it is all t's and no s's in this equation. If the velocity field is static $\mathbf{v}(t, \mathbf{x}) = \mathbf{v}(\mathbf{x})$, then the two equations become the same, and the particle of fluid follows a streamline.

Kinds of flows

- streamline aka laminar vs. turbulent
- compressible vs. incompressible
- viscous vs. inviscous
- rotational vs. irrotational
- steady vs. unsteady aka time-dependent

We will deal mostly with steady, incompressible flow.

Continuity

The current of mass is $\mathbf{j} = \rho\mathbf{v}$. We know that mass is conserved in nonrelativistic fluid flow. What are the consequences? Let's see. Consider a small cube of volume dV=dxdydz. We must have that the flux of mass out of the cube is minus the time rate of change of the mass inside.

time rate of change of mass inside $= -\frac{\partial}{\partial t}(\rho dV) = -\frac{\partial \rho}{\partial t}\, dV$.

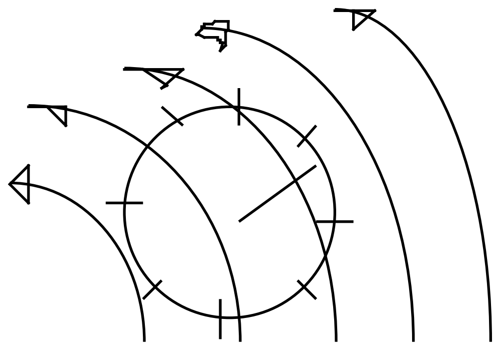
flux out

$$
\begin{aligned}
&= \oint_{\mathbf{S}} d\mathbf{A}\cdot\mathbf{j} \cong j_x(\mathbf{x} + \hat{\mathbf{x}}dx)\,dA - j_x(\mathbf{x})\,dA + (x \to y) + (x \to z) \\
&\cong \left[j_x(\mathbf{x}) + dx\frac{\partial}{\partial x} j_x(\mathbf{x}) \right] dA - j_x(\mathbf{x})\,dA + (x \to y) + (x \to z) \\
&= \frac{\partial}{\partial x} j_x(\mathbf{x})\,dV + (x \to y) + (x \to z) \\
&= \left[\frac{\partial}{\partial x} j_x(\mathbf{x}) + \frac{\partial}{\partial y} j_y(\mathbf{x}) + \frac{\partial}{\partial z} j_z(\mathbf{x}) \right] dV \\
&= \nabla\cdot\mathbf{j}\,dV
\end{aligned}
$$

Thus, we have $\nabla\cdot\mathbf{j} = -\frac{\partial \rho}{\partial t}$. This expresses local conservation of mass. It is called the *continuity equation*.

Circulation

It is convenient to have a measure of the extent to which the flow is going around in circles. This is called the *circulation.* I can test the circulation by putting a little paddle wheel in the fluid. The axis of the paddle wheel is $\hat{\varpi}$. The angular velocity is ϖ . The velocity of a paddle which is at a displacement $\mathbf{r}$ from the axis is $\mathbf{u} = \varpi \times \mathbf{r}$. The velocity of the fluid at the paddle is **v**. The part of **v** that is tangent to the rim of the wheel pushes the wheel around. Let the circle of the paddle wheel be **x**(s). s is arc length. The tangent $\frac{d\mathbf{x}}{ds}$ is a unit vector. The component of **v** along the wheel is $\mathbf{v}\cdot\frac{d\mathbf{x}}{ds}$. The relative speed of the paddle and the tangential part of the fluid flow is $\mathbf{v}\cdot\frac{d\mathbf{x}}{ds} - u$. Suppose that the wheel rotates so that the average of this quantity over the wheel is zero. That gives



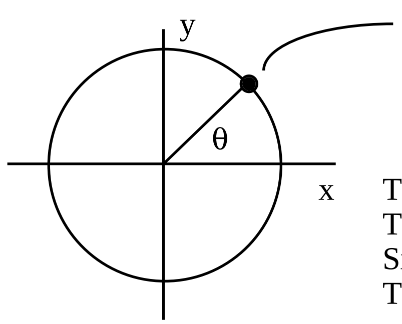
$$\oint_\Gamma \frac{d\mathbf{x}}{ds}\cdot\mathbf{v} = \int ds u$$

$$\oint_\Gamma d\mathbf{x}\cdot\mathbf{v} = 2\pi r u = 2\pi r \varpi r = 2\pi r^2 \varpi$$

The quantity on the LHS is the line integral of the velocity around the curve Γ (which in this case is the circle of the paddle wheel). This line integral $\oint_\Gamma d\mathbf{x}\cdot\mathbf{v}$ is called the *circulation.* It measures the extent to which the fluid is flowing around the path Γ.

Local circulation and the curl of the velocity field

Now let's look at the circulation around a small circle with radius r as r→0 and small area d**A**. The result will be that the small circulation is $dC = d\mathbf{A}\cdot\nabla\times\mathbf{v}$. To get this, we have to assemble a number of pieces. I place the circle in the x,y plane so that d**A** is in the z direction.



(x,y)=(cos θ, sin θ)r   0<θ<2π

$$\frac{d\mathbf{x}}{d\theta} = (-\sin\theta, \cos\theta)r$$

This gives the parameterization of the circle and its tangent. The next step is to look at the velocity field on the circle. Since the circle is small, we can express the velocity in a Taylor expansion about the center. That gives

$$v_x(\mathbf{x}) = v_x(0) + x\frac{\partial}{\partial x}v_x(0) + y\frac{\partial}{\partial y}v_x(0) = v_x(0) + r\left(\cos\theta\frac{\partial}{\partial x}v_x(0) + \sin\theta\frac{\partial}{\partial y}v_x(0)\right)$$

For the y component of v, there is a similar expression with the subscript changed from x to y.

Next we use these expressions for $\mathbf{v}$ and the one above for $\frac{d\mathbf{x}}{d\theta}$ to compute

$$\mathbf{v}\cdot\frac{d\mathbf{x}}{d\theta} = \left(-\sin\theta v_x + \cos\theta v_y\right)r$$

$$= -\sin\theta\left[v_x + \cos\theta\partial_x v_x + \sin\theta\partial_y v_x\right]r^2 + \cos\theta\left[v_y + \cos\theta\partial_x v_y + \sin\theta\partial_y v_y\right]r^2$$

In the second line of this expression, the v's and their derivatives are evaluated at the origin. The notation $\partial_x \equiv \frac{\partial}{\partial x}$ is used. To compute the circulation around the circle, this must be integrated over θ from zero to 2π. Only the terms with $\cos^2\theta$ and $\sin^2\theta$ survive. That gives

$$dC_z = \int_0^{2\pi} d\theta\mathbf{v}\cdot\frac{d\mathbf{x}}{d\theta} = \pi r^2\left(\partial_x v_y - \partial_y v_x\right)$$

This is the circulation around the small circle perpendicular to the z axis. There are similar expressions for circles perpendicular to the x and y axes.

$$dC_x = \int_0^{2\pi} d\theta\mathbf{v}\cdot\frac{d\mathbf{x}}{d\theta} = \pi r^2\left(\partial_y v_z - \partial_z v_y\right)$$

$$dC_y = \int_0^{2\pi} d\theta\mathbf{v}\cdot\frac{d\mathbf{x}}{d\theta} = \pi r^2\left(\partial_z v_x - \partial_x v_z\right)$$

Since the area vector is $d\mathbf{A} = \pi r^2\hat{\mathbf{z}}$ in the first case of the circle of the x,y plane, we have $dC = d\mathbf{A}\cdot\nabla\times\mathbf{v}$. This is also true for an arbitrary orientation of the little circle. This shows that the curl gives the circulation around a small path. The curl of the velocity field is called the *vorticity* $\mathbf{\Omega} = \nabla\times\mathbf{v}$.

To get a little more feel for this, consider the case of a fluid in simple circular motion around the origin with angular velocity ϖ.

Since the speed at radius r is ϖr, $dC = 2\pi r(r\varpi) = 2\varpi\cdot d\mathbf{A}$ but also $dC = d\mathbf{A}\cdot\nabla\times\mathbf{v}$ so $\nabla\times\mathbf{v} = 2\varpi$. This relates the curl to the angular velocity in a simple case.