Flow, etc.

Physics 9HB, Joe Kiskis

Total, local, and convective derivatives

The fluid quantities we are interested in are functions of t and $\mathbf{x}$. Examples are the pressure and the velocity. Let's just use $Q(t, \mathbf{x})$ to stand for any field. If we want to know how Q changes in time *at a fixed point in space*, that's $\frac{\partial Q(t,\mathbf{x})}{\partial t}$, which is called the **local derivative**. However, if we want to know how Q changes *as we ride along with the fluid*, we must think a bit more. The path of an element of the fluid $\mathbf{x}(t)$ satisfies $\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(t, \mathbf{x}(t))$. Thus, if we are riding with the fluid $Q(t) = Q(t, \mathbf{x}(t))$ and

$$\frac{dQ(t)}{dt} = \frac{\partial Q(t,\mathbf{x})}{\partial t} + \sum_{i=1}^{3} \frac{\partial Q(t,\mathbf{x})}{\partial x_i} \frac{dx_i(t)}{dt} \quad (1)$$

$$= \frac{\partial Q(t,\mathbf{x})}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial Q(t,\mathbf{x})}{\partial x_i} \quad (2)$$

$$= \frac{\partial Q(t,\mathbf{x})}{\partial t} + (\mathbf{v} \cdot \nabla) Q(t,\mathbf{x}). \quad (3)$$

The total derivative is the local derivative plus the **convective derivative** (second term).

Derivatives of v

We want to get an equation of motion for the fluid. Newton's second law has the time derivative of the momentum. So we need the time derivative of the momentum of a little element of the fluid in the volume dV. The mass is ρdV, and the velocity is $\mathbf{v}$, so $\mathbf{p}(t, \mathbf{x}) = \rho dV \mathbf{v}(t, \mathbf{x})$. The time rate of change of the momentum of a given element of fluid is the total derivative of $\mathbf{p}$. Since we are restricting ourselves to incompressible flow, we do not need to differentiate ρ, and the derivative of $\mathbf{p}$ is $\rho\times$(the derivative of $\mathbf{v}$). Thus, for each component of $\mathbf{v}$,

$$\frac{dv_i}{dt} = \frac{\partial v_i(t,\mathbf{x})}{\partial t} + (\mathbf{v} \cdot \nabla) v_i(t,\mathbf{x}). \quad (4)$$

If we collect the components together, this can be written

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}(t,\mathbf{x})}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}(t,\mathbf{x}). \tag{5}$$

If the notation of the last term here is not clear, remember that it means just what is in the previous component version.

Equation of motion

The equation of motion for an element of the fluid is $\frac{d\mathbf{p}}{dt} = \mathbf{F}$. Now we know how to write the LHS as

$$\frac{d\mathbf{p}}{dt} = \rho dV[\frac{\partial \mathbf{v}(t,\mathbf{x})}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}(t,\mathbf{x})]. \tag{6}$$

On the RHS, there must be the forces on the element of fluid. We have already seen what these can be. There is a force $\rho dV\mathbf{g}$ due to gravity, and a force $-\nabla P dV$ due to the pressure gradient. There is also a force due to the internal friction of the fluid—viscosity. For now, we will consider the case of negligible viscosity. Later we will add a viscous force. So, after cancelling the dV, that leaves us with

$$\rho[\frac{\partial \mathbf{v}(t,\mathbf{x})}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}(t,\mathbf{x})] = -\nabla P + \rho\mathbf{g} \tag{7}$$

as the equation of motion for inviscous, incompressible flow.

Sometimes it is convenient to write this in a slightly different form. There an *identity* in vector calculus which says

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = (\nabla\times\mathbf{v})\times\mathbf{v} + \frac{1}{2}\nabla v^2. \tag{8}$$

You can check this by tediously writing out the terms on each side and comparing them. Also, the curl of the velocity is called the **vorticity** $\mathbf{\Omega} = \nabla\times\mathbf{v}$. So the alternative form for the equation of motion is

$$\rho[\frac{\partial \mathbf{v}(t,\mathbf{x})}{\partial t} + \mathbf{\Omega}\times\mathbf{v} + \frac{1}{2}\nabla v^2] = -\nabla P + \rho\mathbf{g}. \tag{9}$$

There is another interesting way to write this. To get there, we use $\mathbf{g} = -\nabla\phi$ and a famous identity $\nabla \times \nabla f = 0$. (You can verify it by direct calculation.) Now if we divide by ρ and take the curl of both sides, we get

$$\frac{\partial}{\partial t}\mathbf{\Omega} + \nabla \times (\mathbf{\Omega} \times \mathbf{v}) = -\nabla \times (\frac{1}{\rho}\nabla P). \tag{10}$$

For incompressible flow (ρ = constant), this specializes to the very elegant equation

$$\frac{\partial}{\partial t}\mathbf{\Omega} + \nabla \times (\mathbf{\Omega} \times \mathbf{v}) = 0. \tag{11}$$

This shows that if $\mathbf{\Omega} = 0$ at any time, then it is zero at all times. Vorticity cannot be created in incompressible flow. (Since I *can* stir my coffee, I must wonder about the relevance of inviscous, incompressible flow.)

Incompressible, irrotational flow

There is an especially simple and interesting set of solutions that are irrotational $\mathbf{\Omega} = 0$ and incompressible. In that case, the equations to be solved are $\nabla \times \mathbf{v} = 0$ and $\nabla \cdot \mathbf{v} = 0$. Since $\nabla \times \mathbf{v} = 0$, there exists an H such that $\mathbf{v} = \nabla H$. Then $\nabla \cdot \mathbf{v} = \nabla^2 H$, and the equation to solve is $\nabla^2 H = 0$. This is called potential flow because the electrostatic potential that gives rise to the electric field solves the same equation.

Bernoulli equation

The Bernoulli equation applies to incompressible, steady flow. Use $\partial\mathbf{v}/\partial t = 0$ and $\mathbf{g} = -\nabla\phi$, then dot $\mathbf{v}$ into the equation of motion. Since $\mathbf{v} \cdot \mathbf{\Omega} \times \mathbf{v} = 0$, we get

$$\rho(\mathbf{v} \cdot \nabla)(\frac{1}{2}v^2) = -\mathbf{v} \cdot \nabla(P + \rho\phi) \tag{12}$$

or

$$\mathbf{v} \cdot \nabla(\frac{1}{2}\rho v^2 + P + \rho\phi) = 0. \tag{13}$$

$\mathbf{v} \cdot \nabla$ is the derivative in the direction of $\mathbf{v}$. Thus, along a streamline,

$$\frac{1}{2}\rho v^2 + P + \rho\phi = constant. \tag{14}$$

This constant may vary from streamline to streamline. This is **Bernoulli's equation**.

For irrotational flow, there is a stronger result. In that case, the equation of motion becomes simply

$$\nabla(\frac{1}{2}v^2) = \nabla(P + \rho\phi) \tag{15}$$

so

$$\nabla(\frac{1}{2}v^2 + P + \rho\phi) = 0 \tag{16}$$

and

$$\frac{1}{2}\rho v^2 + P + \rho\phi = constant. \tag{17}$$

Now the constant is the same for all streamlines, i.e. it's independent of space and time. However, it does depend upon the problem at hand and must be determined from some given data.