Vector calculus: grad, curl, div, and all that

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1 Introduction

This is for your reference. As the need for these techniques arises in the physics, they will be introduced. Also, you will be studying similar material in Math 21D. You will use these techniques very heavily next quarter in Physics 9C or 9HC.

Caution: This handout and others that may follow are **notes**—not a polished product. They will be a bit terse, and surely some errors will creep in. I hope the notes are helpful. Your comments are welcome.

2 Fields

The concept of a field especially a vector field is one of the great ideas in physics. You have probably already studied one example of this: the gravitational field. Newton's universal law of gravitation gives the force of a mass M at the origin on a mass m at position $\mathbf{r}$ as $\mathbf{F} = GMm\mathbf{r}/r^3$. How does M exert a force on m when it is a distance r away? This is the action-at-a-distance problem. It can be overcome by introducing the concept of the gravitational field. At first, it seems like a mere linguistic trick. However, it turns out to be one of the most useful general ideas in physics. We say that the mass M creates a gravitational field that fills all space and that the the field at $\mathbf{r}$ acts on the mass m at $\mathbf{r}$. The field is $\mathbf{g}(\mathbf{r}) = GM\mathbf{r}/r^3$ and the force is $\mathbf{F} = m\mathbf{g}(\mathbf{r})$. Next quarter you will study the greatest examples of fields: the electric and the magnetic fields. This quarter the examples are more concrete.

We begin with the study of fluids. Last quarter much attention was given to the motion of particles or of objects that could be approximated as point particles. The configuration of a particle is entirely specified by its position $\mathbf{r}$. Since fluids are systems of particles, we could specify the configuration of the fluid by labeling all the particles and then giving the position of each. This is not a very good way to go. Since we can't see the paths of the individual particles much less the imagined little labels, there is not much use for such a description. It is more productive to describe what the fluid is doing at a particular point $\mathbf{x}$ in space. There are various properties of the fluid at position $\mathbf{x}$ that are interesting to us. The pressure P is one example. This is a function defined on the three-dimensional space of positions. The values that it takes are in the one-dimensional space of pressures—the real line. This is what we call a **scalar field**.

The main thing we will be interested in is the velocity or the fluid. Thus, the function $\mathbf{v}(\mathbf{x})$, which gives the velocity of the fluid at each point $\mathbf{x}$, will be the way that we specify the configuration of the fluid.

Consider $\mathbf{v}(\mathbf{x})$. This is a function defined on the three-dimensional space of positions. The values that it takes are in the three-dimensional vector space of velocities. This is what we call a **vector field**.

To describe the motion of the fluid, we will have some equations of motion. Since the fluid is really made of a bunch of particles, and those particles obey Newton's equations, we can expect some differential equations somehow related to $\mathbf{F} = d\mathbf{p}/dt$. Thus, we are going to need to be able to deal with the calculus of scalar and vector *fields*. When we include the time dependence, the fields are functions of four variables, e.g. $P(t, \mathbf{x})$. We can differentiate the t dependence in P: $\frac{\partial P}{\partial t}$, the x dependence $\frac{\partial P}{\partial x}$, the y dependence, or the z dependence.

Note that it is sometimes convenient to call (x, y, z) (x_1, x_2, x_3). Also the shorthand $\partial_t = \frac{\partial}{\partial t}$, $\partial_x = \partial_1 = \frac{\partial}{\partial x}$, etc. is useful.

3 Gradient

In vector calculus, there are various new spatial derivatives. The time derivative will reappear when we get to the equation of motion. The rate of change of the pressure in the x-direction is $\partial_x P$. Suppose we want the rate of change in some arbitrary direction $\hat{\mathbf{a}} = (a_1, a_2, a_3)$. That's

$$\frac{d}{ds} P(\mathbf{x} + s\hat{\mathbf{a}})|_{s=0} = a_1 \partial_1 P(\mathbf{x}) + a_2 \partial_2 P(\mathbf{x}) + a_3 \partial_3 P(\mathbf{x}). \tag{1}$$

If we introduce a vector field $\nabla P(\mathbf{x})$ defined by

$$\nabla P(\mathbf{x}) = \hat{\mathbf{x}}_1 \partial_1 P(\mathbf{x}) + \hat{\mathbf{x}}_2 \partial_2 P(\mathbf{x}) + \hat{\mathbf{x}}_3 \partial_3 P(\mathbf{x}) \tag{2}$$

then we can write

$$\frac{d}{ds}P(\mathbf{x}+s\hat{\mathbf{a}})|_{s=0} = \hat{\mathbf{a}}\cdot\nabla P(\mathbf{x}) \tag{3}$$

for the rate of change of P in the direction $\hat{\mathbf{a}}$. This useful object $\nabla P(\mathbf{x})$ is called the **gradient** of P. Evidently, it is a vector field.

4 Divergence

Now consider the flow of the fluid. It is described by the velocity field $\mathbf{v}(\mathbf{x})$. There is another scalar field: the density $\rho(\mathbf{x})$. With that we can construct $\mathbf{j}(\mathbf{x}) = \rho(\mathbf{x})\mathbf{v}(\mathbf{x})$. This is the current of fluid mass. Consider a small surface element $d\mathbf{A}$ at $\mathbf{x}$. The flux of mass (mass per unit area per unit time) through $d\mathbf{A}$ is $\mathbf{j}\cdot d\mathbf{A}$. Let's look at the flux of mass out of a small cube with one corner at $\mathbf{x}$ and extending a distance dx down the x-axis, etc. The flux out of the face with surface element $d\mathbf{A} = \hat{\mathbf{x}}dxdy$ at $\mathbf{x}+\hat{\mathbf{x}}dx$ is approximately ($dx \to 0$, etc.)

$$\begin{aligned}
\mathbf{j}(\mathbf{x}+\hat{\mathbf{x}}dx)\cdot\hat{\mathbf{x}}dxdy &= j_x(\mathbf{x}+\hat{\mathbf{x}}dx)dydz & (4)\\
&= [j_x(\mathbf{x}) + \partial_x j_x(\mathbf{x})dx]dydz & (5)\\
&= j_x(\mathbf{x})dydz + \partial_x j_x(\mathbf{x})dxdydz. & (6)
\end{aligned}$$

The flux *out* of the cube through the parallel face at the origin is $\mathbf{j}(\mathbf{x})\cdot(-\hat{\mathbf{x}}dxdy)$. The net flux out of the cube through these two parallel faces is $\partial_x j_x(\mathbf{x})dxdydz$. To get the total flux out of the cube, we must include the other two directions. We get $(\partial_x j_x + \partial_y j_y + \partial_z j_z)dxdydx$. If we think of the object

$$\nabla = \hat{\mathbf{x}}_1\partial_1 + \hat{\mathbf{x}}_2\partial_2 + \hat{\mathbf{x}}_3\partial_3 \tag{7}$$

as a vector operator, then we can introduce the nice combination

$$\nabla\cdot\mathbf{j} = \partial_1 j_1 + \partial_2 j_2 + \partial_3 j_3 \tag{8}$$

and write the flux out of the small cube as $\nabla\cdot\mathbf{j}dxdydz$. The combination $\nabla\cdot\mathbf{j}$ is a scalar field that is called the **divergence** of the vector field $\mathbf{j}$.

If there are no sources or sinks of material and the mass cannot disappear and we must have

$$\begin{aligned}
\nabla\cdot\mathbf{j}(t,\mathbf{x})dxdydz &= -\partial_t\,(\textit{mass in the cube}) & (9)\\
&= -\partial_t[\rho(t,\mathbf{x})dxdydz]. & (10)
\end{aligned}$$

The relation $\nabla \cdot \mathbf{j} = -\partial_t \rho$ is called the **continuity equation**. It expresses the *conservation of mass.*

5 Curl

Next we come to the most difficult one: the **curl** of a vector field. It is a local (at a point $\mathbf{x}$) measure of the extent to which the vector field is circulating. Think of whirlpools or something like that. In fact, let's start with the definition of the **circulation** on a closed curve Γ of the vector field $\mathbf{v}(\mathbf{x})$. The curve is parameterized with s from 0 to 1, is written $\mathbf{x}(s)$, and has tangent or "velocity" $d\mathbf{x}(s)/ds$. (Caution: Do not confuse this "velocity" associated with the curve Γ with the velocity field $\mathbf{v}$.)

$$C = \oint_\Gamma \mathbf{v} \cdot d\mathbf{x} = \int_0^1 \mathbf{v}(\mathbf{x}(s)) \cdot \frac{d\mathbf{x}(s)}{ds} ds. \tag{11}$$

If Γ goes around with a whirlpool in $\mathbf{v}$, C will be large. If Γ is in a region where $\mathbf{v}$ is a constant, uniform flow, $C = 0$.

To get to the curl, consider a small rectangular path in the x, y-plane. Its normal is $\hat{\mathbf{z}}$ so $d\mathbf{A} = \hat{\mathbf{z}} dx dy$. The associated circulation is

$$\begin{aligned} dC_z &= \int_0^{.25} v_x(\mathbf{x} + 4s dx \hat{\mathbf{x}}) ds + \int_0^{.25} v_y(\mathbf{x} + dx\hat{\mathbf{x}} + 4s dy \hat{\mathbf{y}}) ds & (12) \\ &- \int_0^{.25} v_x(\mathbf{x} + dy\hat{\mathbf{y}} + 4s dx \hat{\mathbf{x}}) ds - \int_0^{.25} v_y(\mathbf{x} + 4s dy \hat{\mathbf{y}}) ds & (13) \\ &\approx dx dy [\partial_x v_y(\mathbf{x}) - \partial_y v_x(\mathbf{x})]. & (14) \end{aligned}$$

Use the Taylor expansion repeatedly to fill in the missing steps! Now I'm sure you can guess what we would get if we did the same thing for little surfaces with normals $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$. Let's collect these together in a vector called the curl of $\mathbf{v}$.

$$\nabla \times \mathbf{v}(\mathbf{x}) = \hat{\mathbf{x}}[\partial_y v_z(\mathbf{x}) - \partial_z v_y(\mathbf{x})] + \hat{\mathbf{y}}[\partial_z v_x(\mathbf{x}) - \partial_x v_z(\mathbf{x})] + \hat{\mathbf{z}}[\partial_x v_y(\mathbf{x}) - \partial_y v_x(\mathbf{x})]. \tag{15}$$

Then we have $dC = d\mathbf{A} \cdot \nabla \times \mathbf{v}$. The quantity $\mathbf{\Omega} = \nabla \times \mathbf{v}$ is called the **vorticity**.

6 Laplacian

The last thing to discuss is the **Laplacian**. It is a second order operator. Suppose the vector field $\mathbf{v}$ were the gradient of a scalar $V(\mathbf{x})$: $\mathbf{v} = \nabla V$. What is the divergence of $\mathbf{v}$? Using the definitions above, we get

$$\nabla \cdot \mathbf{v} = \nabla \cdot (\nabla V) = (\partial_x^2 + \partial_y^2 + \partial_z^2)V. \tag{16}$$

This operator called the Laplacian $\nabla \cdot \nabla = \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ appears in many problems.

Good. Done. Now you know it all!