Wave equation

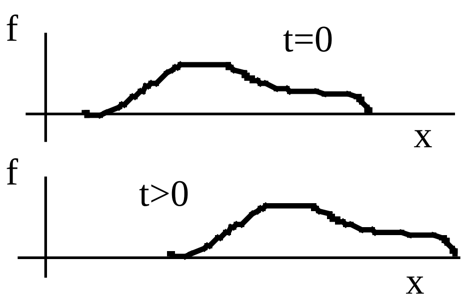
The bottom line of this little story is the *wave equation*

$$\frac{\partial^2 f(x,t)}{\partial t^2} = v^2 \frac{\partial^2 f(x,t)}{\partial x^2} \ .$$

This is the differential equation that describes the propagation of *dissipationless*, *dispersionless* waves. This derivation will be inductive and general. For each specific case like tension waves on a string or sound waves in the air, it is also possible to give a detailed deductive derivation that applies to that physical system.

We begin with the observation of waves on the hose or on the rods. We see that to a fair approximation, they have two properties. First the waves do not change size as they propagate. Actually they do slowly get smaller with time, but we can imagine doing a better job of eliminating the "friction" so that they approach the limit of propagating on indefinitely without getting smaller. This is the idealization of waves without dissipation. Second we see two other properties that are equivalent. The waves keep the same shape as they propagate, and all shapes have the same speed. Waves with this property are said to have no dispersion. (For media with dispersion, waves of different wavelength have different speeds. An example is light in glass. That is why a prism can separate the colors of the rainbow in white light.) Our demonstration examples are a better approximation to the idealization of no dispersion than they are to that of no dissipation.

This means that if we know the shape of the wave at one time, then we know it at later times by just moving it over by a distance vt.



This can be expressed in equations as follows: Suppose that at t=0, the shape of the wave is given by a functions a(x) so that f(x,0) = a(x). What is f(x,t) at a later time? Since it just gets moved over by vt, we have f(x,t) = f(x-vt,0) = a(x-vt). Why you ask is the argument x-vt. Suppose the function a(x) has peak at $x=x_0$. Then with the form above, the peak will be where $x-vt=x_0$ or $x=x_0+vt$, *i.e.* the peak moves to the right with speed v. All this is independent of the actual form of a(x). We assert that the equation we are looking for must have the property that for any function a(·), a(x-vt) is a solution. But we don't want to be stuck with waves going to the right only. Thus we also insist that for any b(·), a left-moving wave of the form b(x+vt) must also be a solution. With this input, we can figure out what the form of the differential equation describing the waves must be.

$$\frac{\partial f}{\partial x} = a'(x - vt) \qquad \frac{\partial^2 f}{\partial x^2} = a''(x - vt)$$

$$\frac{\partial f}{\partial t} = -va'(x - vt) \qquad \frac{\partial^2 f}{\partial t^2} = v^2 a''(x - vt)$$

so that $$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$ as advertised.

A similar calculation with a(x-vt) replaced by b(x+vt) gives the same final equation for f(x,t). (You might ask why I did not stop with the first order relation $\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x}$. The answer is that that would have restricted me to right moving waves only with a(x-vt) a solution but not b(x+vt).) The quantity v is not determined in this approach. The speed of the waves is a characteristic of the particular medium at hand and needs to be determined from a detailed understanding of the medium.

An important property of the wave equation is that it is linear (more precisely: homogeneous of degree one in f). That is equivalent to the *superposition principle*. You can easily check that if f_1 and f_2 are solutions to the wave equation, then any linear combination of those two functions is also a solution.

The most general solution to the wave equation is a sum of an arbitrary right-moving wave and an arbitrary left-moving wave: f(x,t) = a(x-vt) + b(x+vt). This is a *lot* of solutions. How is the solution of interest in any particular case singled out? You have already encountered a similar situation. The differential equation of Newton's second law, F=ma also has lots of solutions, and the right one for any particular case is determined by the initial conditions of position and velocity. For the wave equation, the situation is analogous. Given the initial configuration of the string f(x,0) and the initial rate of change of the displacement $\frac{\partial f(x,0)}{\partial t}$, the functions a and b can be determined, and that gives the subsequent motion for all time.

For example, suppose that at t=0, the displacement is given by a function F(x) so that f(x,0) = F(x). Suppose also that at t = 0, the string is not moving so that $\frac{\partial f(x,0)}{\partial t} = 0$. Then with a little fiddling around to fix a and b, one finds that the solution for all time is f(x,t) = [F(x-vt) + F(x+vt)]/2. You should be able to check that this is a solution and that it has the right initial conditions. What does it tell you?

That's all great if you are stuck in just one spatial dimension. In the real world, we have three spatial dimensions. How does this generalize? It turns out that it is about as simple as it could be. The function that describes the disturbance is now a function of x, y, z, and t, f(x,y,z,t). The three-dimensional wave equation is

$$\frac{\partial^2 f}{\partial t^2} = v^2 \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right]$$

The combination of partial derivatives that appears on the right hand side comes up in many places, so to save writing, it gets its own notation

$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Then the wave equation in three dimensions can be written more easily as

$$\frac{\partial^2 f}{\partial t^2} = v^2 \nabla^2 f \ .$$

It is not quite so easy to find solutions to this, but here are a couple that you can check. Let $\mathbf{x}$ be the position three-vector, and let $\mathbf{k}$ be a fixed vector. then

$f = A\sin(\mathbf{k}\cdot\mathbf{x} - \omega t)$ is a solution if ω and the magnitude of the vector $\mathbf{k}$ are correctly related. What is the relation? This kind of solution is called a *plane wave*. Describe it. Another possibility is a spherical wave

$$f = \frac{A}{r}\sin(kr - \omega t) \quad \text{with} \quad r = |\mathbf{x}| \quad \text{and k a constant.}$$

Of course that are many other possibilities.

Summary

Dissipationless, dispersionless waves in one dimension are solutions to the wave equation. $\frac{\partial^2 f(x,t)}{\partial t^2} = v^2 \frac{\partial^2 f(x,t)}{\partial x^2}$. It embodies the superposition principle so that if f_1 and f_2 are solutions then so are all linear combinations of f_1 and f_2. The general solution has the form f(x,t) = a(x-vt) + b(x+vt).