Classical groups

These are Lie groups that are subgroups of $\text{GL}(n,\mathbb{C})$.

Special groups

$n$ vectors in an $n$-dimensional vector space determine a volume: $x_i = e_j x_i^j$ $i = 1, ..., n$ gives the volume $v = \epsilon_{i_1...i_n} x_1^{i_1} ... x_n^{i_n}$. After a transformation, there are $n$ new vectors $x'_i = e_k D_{ij} x_i^j$ and a new volume $v' = \epsilon_{i_1...i_n} D_{j_1}^{i_1} ... D_{j_n}^{i_n} x_1^{i_1} ... x_n^{i_n}$.

To get $v = v'$ for any set of vectors $x_i$, we need $\epsilon_{i_1...i_n} D_{j_1}^{i_1} ... D_{j_n}^{i_n} = \epsilon_{j_1...j_n}$ or $\det D = 1$. Transformations satisfying this condition form a group—the special linear group $\text{SL}(n,\mathbb{C})$.

Metrics

For a more refined structure, a metric is introduced. $g(x', x) \in C$ or $R$ for two vectors $x$ and $x'$. $g$ is nonsingular and linear on $x$ $g(x', ax_1 + bx_2) = ag(x', x_1) + bg(x', x_2)$.

Bilinear metrics are also linear on $x'$ $g(ax_1' + bx_2', x) = ag(x_1', x) + bg(x_2', x)$.

Sesquilinear metrics are still linear on $x$, but on $x'$ $g(ax_1' + bx_2', x) = a^* g(x_1', x) + b^* g(x_2', x)$.

Bilinear symmetric: bilinear and $g(x', x) = g(x, x')$.

Sesquilinear symmetric: sesquilinear and $g(x', x) = g(x, x')^*$.

Bilinear antisymmetric: bilinear and $g(x', x) = -g(x, x')$.

Orthogonal groups

The field is the reals. For a bilinear symmetric metric, we can find a basis with $g_{ij} = g(e_i, e_j) = \text{diag}(1, ..., 1, -1, ..., -1)$ with $n_+$ 1’s and $n_-$ -1’s. The orthogonal transformations $\mathcal{O}(n_+, n_-, R)$ are those that preserve that form of the metric so that $g_{ij} = g_{kl} D_{ik}^l D_{jl}^k$. The further requirement that $\det D = 1$ gives the special orthogonal groups $\text{SO}(n_+, n_-, R)$.

Unitary groups

Now the field is the complexes. A sesquilinear symmetric can also be put in the same diagonal form. Then the transformations satisfying $g_{ij} = g_{kl} D_{ik}^l D_{jl}^k$ form the unitary group $\mathcal{U}(n_+, n_-)$ and with $\det D = 1$, the special unitary group $\text{SU}(n_+, n_-)$. 

Symplectic groups

The field can be R or C. The bilinear antisymmetric metrics can be nonsingular for even n and satisfy $g_{ji} = -g_{ij}$. The symplectic groups $Sp(n, R$ or $C)$ preserve a metric with this property. A common choice for $g$ is

$$
\begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -1 & \ldots & 0 & 0 \\
-1 & 0 & \ldots & 0 & 0
\end{pmatrix}
$$

(1)