Physics 223B, Joe Kiskis

Key facts

- SU(2)'s
- If α is a root, then $n\alpha$ is not a root unless $n = 0, \pm 1$.
- For SU(3), we got the simple roots fro the explicit Gell-Mann matrices. More generally, the possible roots come from the classification theorem of all possible simple Lie algebras.
- Master equation:
$$\frac{2\alpha \cdot \beta}{\alpha^2} = q - p \tag{1}$$
- Simple roots (positive roots not the sum of other positive roots)
- If α and β are simple roots, $\alpha - \beta$ and $\beta - \alpha$ are not roots.
- The number of simple roots is m, the rank of the algebra. They are linearly independent and complete, but not necessarily orthonormal.
- All positive roots are of the form $\sum_\alpha k_\alpha \alpha$. The sum is over the simple roots, and the k_α are non-negative integers.
- The positive roots are built from all possible sums of the simple roots consistent with the master equation. It helps to exploit the Weyl symmetry, the SU(2)'s, and the fact the in the adjoint irrep, the roots are all in "$j = 1$" irreps of the SU(2)'s in their own direction. There are also the m zero "roots" at the center associated with the m diagonal generators.
- The missing commutators are $[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$. The $N_{\alpha\beta}$ are determined by using the "known" raising and lowering rules for SU(2). Each root is in an SU(2) irrep of each of the SU(2)'s,

Fundamental highest weights and fundamental irreps:

- μ is a highest weight iff $E_\alpha |\mu\rangle = 0$ for each simple root α.
- $$\frac{2\alpha^j \cdot \mu}{(\alpha^j)^2} = q - p = q = l^j \geq 0 \tag{2}$$
- The $(l^1, l^2, \ldots, l^m)$ and the components of μ give equivalent information. They specify the (highest) weight vector μ.

- The m fundamental highest weights are those that have exactly one $l^j = 1$ and the rest of the l's equal to zero. If the 1 is in position j, the weight is called μ^j So the highest fundamental weights μ^j are specified by their list of l's $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with exactly one non-zero entry in the jth position.

- Any highest weight is then of the form $\mu = \sum_j l^j \mu^j$ with the l^j any collection of non-negative integers.

- In tensor products, the highest weights add, so any irrep μ can be found in the product of the form $\mu^1 \times \ldots \times \mu^1 \times \mu^2 \times \ldots \times \mu^2 \ldots$ with each μ^j appearing l^j times.

- All of the weights of an irrep are of the form $E_{-\alpha} E_{-\beta} \ldots |\mu\rangle$ with simple $\alpha, \beta, \ldots$. Thus one lowers away from the highest weight in all possible ways consistent with the values of the p's and q's. Again, a huge amount of work is saved by exploiting Weyl symmetry. If there is more than one route to a weight, degeneracy needs to be checked.