Physics 223B, winter 2014

1 Classes and cosets of the Euclidian group E_2

This is a summary some properties of E_2, some of which we covered and some that we didn't get to for lack of time.

The Euclidean group E_2 is all the rigid transformations of the plane R^2 connected to the identity, *i.e.* it leaves out the reflections. We will study this in great detail later. Let T be subgroup of E_2 that is all the translations of the plane R^2. A general element is denoted t. Let R be the subgroup of rotations about the origin. A general element is r. You can get all of the elements of E_2 from products of elements from these two subgroups. Indeed the most general element is a rotation followed by a translation. If that assertion is seems questionable to you now, it might be less so by the end of this discussion.

1.1 Invariant subgroup

Are either T or R invariant? trt^{-1} is not a rotation about the origin, *i.e.* it is not an element of R. It is a rotation by the same angle about the translated origin. Thus R is not invariant. On the other hand, rtr^{-1} is a translation by the same distance as t but in a direction rotated from t by r, so T is invariant.

1.2 Classes

From the last statement above, it follows that the class of a non-zero translation is all the translations by the same distance. Referring to the statement that trt^{-1} is a rotation by the same angle about the translated origin, we conclude that the class of a rotation is all the rotations by the same angle about any point.

1.3 Cosets

The cosets of R are of the form tR. You can show that if t and t' are distinct, then so are the corresponding cosets. The coset tR is all the transformations that carry the origin o to the point to, and E_2/R is isomorphic to R^2. In this way of thinking about it, R is the isotropy subgroup of a point and the coset is the space itself. In this way, we construct space from a group by moding out the isotropy subgroup.

The cosets of T are of the form rT. The is an arbitrary translation followed by a particular rotation about the origin. These are all the transformations that change the angle of the x-axis with itself by the rotation r.

Since T is invariant, the cosets E_2/T are a subgroup of E_2. Let $l(\theta)$ be the object that is the collection of all the lines that make an angle θ with the x-axis, and let L be the collection of the $l(\theta)$ for all θ. You can convince yourself that

$tl(\theta) = l(\theta)$ and $r(\phi)l(\theta) = l(\theta + \phi)$. Thus the coset rT acts as a rotation on L. This is an explicit construction of the abstract statement that $E_2/T = R$.