Main points, Chapter 3, representations

Vector space V.
Linear operators GL(V): V $\to$ V
Group G
Representation of G: homomorphism $g \in G \to U(g) \in GL(V)$

  U(g): V $\to$ V

Equivalence:

  S: V $\to$ V′
  $U' = SUS^{-1}$

Character: $\chi(g) = \mathrm{Tr}\, U(g)$ (Constant on classes. Same for equivalent reps.)

Basis in V: e_i , i=1, …, dim V
Matrix elements of U(g): $U(g)\, e_i = e_j\, D^j{}_i(g)$
Reducibility, etc.

  Subspace: $V_1 \subset V$
  Invariant subspace: $v_1 \in V_1 \to U(g)v_1 \in V_1\ \forall v_1, g$ I.e. $UV_1 \subset V_1$
  If there is such a non-trivial V_1 , then we say that U is **reducible**.
  If there is a linearly independent V_2 with $V = V_1 \oplus V_2$ and V_2 is also invarient, then U is **fully reducible = decomposable**.

  There is a basis with $D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$

  If there are no non-trivial invariant subspaces, then U is **irreducible**.
  For compact groups, all representations are equivalent to **unitary** representations.
  For unitary representations, reducible $\Rightarrow$ fully reducible.

Tools: Schurs's lemmas.

  Irreps of an abelian group are of dimension one.

Orthonormality and completeness of representation matrices for irrep:

  $D^{\mu}(g)^i{}_j$

  μ ranges over the inequivalent irreps.
  n_μ is the dimension of the irrep μ
  i and j run from 1 to n_μ
  The order of G is n_G

  Think of the D's as a set of vectors labeled by the collection (μ, i, j) and with components labeled by g.
  Then the vectors $\sqrt{\frac{n_\mu}{n_G}} D$ are orthonormal and complete.
  The character on the class i in the irrep μ is χ^μ_i .
  The normalized characters $\tilde{\chi}^\mu_i = \sqrt{\frac{n_i}{n_G}} \chi^\mu_i$ are also orthonormal and complete.

Number of irreps = number of classes.

The irrep μ appears in the decomposition of a rep $\tilde{\chi}_{\mu}^{*\,i}\chi_i$ times.

A rep is irreducible iff $\tilde{\chi}^{*\,i}\chi_i = 1$