Physics 245A, winter 2006, Joe Kiskis

Notes Lie groups

Classical groups

All the nonsingular, *i.e.* invertable, linear transformations of an n-dimensional complex vector space form a group called the general linear group GL(n,C). These are Lie groups. The *classical groups* are Lie groups that are subgroups of the GL(n,C).

Special groups

Let e_j with $j = 1, ..., n$ be n basis vectors for an n-dimensional vector space. A general vector x with components x^j in this basis is written $x = e_j x^j$ (implied sum on j). n such vectors $x_i = e_j x_i^j$ $i = 1, ..., n$ determine a volume $v = \epsilon_{i_1 \dots i_n} x_1^{i_1} ... x_n^{i_n}$. After a transformation, there are n new vectors $x_i' = e_k D_j^k x_i^j$ and a new volume $v' = \epsilon_{i_1 \dots i_n} D_{j_1}^{i_1} x_1^{j_1} ... D_{j_n}^{i_n} x_n^{j_n}$. To get $v = v'$ for any set of vectors x_i, we need $\epsilon_{i_1 \dots i_n} D_{j_1}^{i_1} ... D_{j_n}^{i_n} = \epsilon_{j_1 \dots j_n}$ or $\det D = 1$. Transformations satisfying this condition form a group—the special linear group SL(n,C).

Metrics

For a more refined structure, a metric is introduced. $g(x', x) \in C \ or \ R$ for two vectors x and x'. g is nonsingular and linear on x $g(x', ax_1 + bx_2) = ag(x', x_1) + bg(x', x_2)$.
Bilinear metrics are also linear on x' $g(ax_1' + bx_2', x) = ag(x_1', x) + bg(x_2', x)$.
Sesquilinear metrics are still linear on x, but on x' $g(ax_1' + bx_2', x) = a^* g(x_1', x) + b^* g(x_2', x)$.
Bilinear symmetric: bilinear and $g(x', x) = g(x, x')$.
Sesquilinear symmetric: sesquilinear and $g(x', x) = g(x, x')^*$.
Bilinear antisymmetric: bilinear and $g(x', x) = -g(x, x')$.

Orthogonal groups

The field is the reals. For a bilinear symmetric metric, we can find a basis with $g_{ij} = g(e_i, e_j) = diag(1, ..., 1, -1..., -1)$ with n_+ 1's and n_- -1's. The orthogonal transformations $O(n_+, n_-, R)$ are those that preserve that form of the metric so that $g_{ij} = g_{kl} D^k_i D^l_j$. The further requirement that $\det D = 1$ gives the special orthogonal groups $SO(n_+, n_-, R)$.

Unitary groups

Now the field is the complexes. A sesquilinear symmetric can also be put in the same diagonal form. Then the transformations satisfying $g_{ij} = g_{kl} D^{k*}_i D^l_j$ form the unitary group $U(n_+, n_-)$ and with $\det D = 1$, the special unitary group $SU(n_+, n_-)$.

Symplectic groups

The field can be R or C. The bilinear antisymmetric metrics can be nonsingular for even n and satisfy $g_{ji} = -g_{ij}$. The symplectic groups $Sp(n, R\ or\ C)$ preserve a metric with this property. A common choice for g is

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & -1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{1}$$