

# Physics 245A, winter 2006, Joe Kiskis

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## Lie groups and Lie algebras

Topologically the discrete groups are sets of points that can be labeled with integers. Now we study continuous groups like  $SO(2)$  and  $SO(3)$  or the classical matrix groups. These groups are manifolds topologically: locally they look like pieces of  $R^n$  for some  $n$ . The elements of the group can be labeled with a set of  $n$  real numbers. For  $SO(2)$ ,  $n = 1$ ; for  $SO(3)$ ,  $n = 3$ ; for  $SO(N)$ ,  $n = N(N - 1)/2$ . We want to relate group structure and manifold structure, i.e. algebra and topology. The local manifold structure is seen in the tangent vector space. Since any region of the group can be mapped to a region around the identity  $e$ , by a group multiplication, we can concentrate on the area around  $e$ .

The tangent space at the  $e$  is the *Lie algebra* of the group. It is a vector space.

Consider a matrix group and put on coordinates  $x_i$   $i = 1, \dots, n$ . The elements of the group are then matrices  $D(x)$ . There are many ways to choose the coordinates. Let the identity have all coordinates zero  $D(0) = e$ . Look at the region around the identity.

$$D(x) = e + x_i \frac{\partial D}{\partial x_i} \Big|_{x=0} + \frac{1}{2} x_i x_j \frac{\partial^2 D}{\partial x_i \partial x_j} \Big|_{x=0} \quad (1)$$

For unitary  $D$ ,  $\frac{\partial D}{\partial x_i} \Big|_{x=0}$  is an antihermitian matrix. Define a hermitian matrix  $T_j \equiv i \frac{\partial D}{\partial x_j} \Big|_{x=0}$  so that

$$D(x) = e - iT_j x^j + O(x^2). \quad (2)$$

For different choices of coordinates, there will be different  $T$ 's. Different sets will be linear combinations of each other. The Lie algebra is the  $n$ -dimensional vector space with basis  $\{T_i, i = 1, \dots, n\}$  and vectors  $T_i x^i$ .

Each element of the group near the identity gives an element of the algebra. A change coordinates on the group  $G$  gives a change of basis in the algebra  $\mathcal{G}$ .

So far,  $\mathcal{G}$  is just a vector space. What makes it an algebra? That is some multiplication  $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ . For  $SO(3)$ ,  $T_i = J_i$  and the multiplication of basis vectors is

$$[J_i, J_j] = i\epsilon_{ijk} J_k. \quad (3)$$

For general vectors  $a = J_i a^i$  and  $b = J_i b^i$ ,

$$[a, b] = [J_i, J_j] a^i b^j = J_k (i\epsilon_{ijk} a^i b^j) = i(a \times b) \cdot J. \quad (4)$$

Now it is easy to generalize this to any matrix group. The  $T$ 's are matrices so that we can define the multiplication of general vectors  $a = T_i a^i$  and  $b = T_j b^j$  by

$$[a, b] = [T_i, T_j] a^i b^j. \quad (5)$$

But how do we know this will work? How do we know that the RHS gives another element of the algebra? It is ok if

$$[T_i, T_j] = i c_{ij}{}^k T_k. \quad (6)$$

To show this, consider the commutator at the level of the group elements

$$D(z) = D^{-1}(y) D^{-1}(x) D(y) D(x) \quad (7)$$

and calculate it to second order in  $x$  and  $y$ . The result is

$$D(z) = 1 - [T \cdot y, T \cdot x]. \quad (8)$$

By continuity and the closure of the *group* multiplication, this is an element of the group near the identity  $1 - iT \cdot z$ . Thus

$$[T \cdot y, T \cdot x] = iT \cdot z. \quad (9)$$

Thus the *algebra* multiplication rule, as defined above, does close. Further, if we know the group multiplication rule, then we know  $z$  in terms of  $x$  and  $y$ . Due to the antisymmetric and bilinear structure of (9), it must be that  $z^k = c_{ij}{}^k y^i x^j$  with  $c$  antisymmetric in  $ij$ . Thus, (6) is established.

We see that the commutator in the algebra is determined by the group multiplication through the commutator in the group. It measures the failure of the box to close.

Conversely, if we know the commutator in the algebra, we can determine the group near the origin using

$$e^A e^B = e^C \quad (10)$$

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [B, A]] + \dots \quad (11)$$

The structure constants  $c_{ij}{}^k$  are basis specific like a metric is. The commutator is the multiplication that makes the Lie algebra into an algebra.