Power (energy transport)

Waves have energy in the distortion and the movement of the medium. Since the disturbance moves, the energy is transported. The energy transported per unit time is the power of the wave.

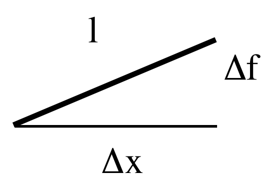
Kinetic energy

A little element of the string that is moving up or down has kinetic energy $mu^2/2$. If the element has length Δx, and the mass per unit length is μ, then $m = \mu\Delta x$. The transverse velocity is $u = \partial f/\partial t$. Thus the kinetic energy per unit length is

$$\mu(\partial f/\partial t)^2/2.$$

Potential energy

Consider a length Δx along the x-axis. If the slope of the string is not zero there, then the element of string that started out with length Δx has been stretched to a longer length.



$$l = \sqrt{\Delta x^2 + \Delta f^2} = \Delta x\sqrt{1 + (\frac{\Delta f}{\Delta x})^2} \approx \Delta x\left[1 + \frac{1}{2}\left(\frac{\partial f}{\partial x}\right)^2\right] = \Delta x + \Delta x\,\frac{1}{2}\left(\frac{\partial f}{\partial x}\right)^2$$

The change in the length times the force, which is the tension in the string F_T , is the work done in stretching the element of string and is thus the potential energy stored in the element. So the potential energy per unit length is

$$F_T(\partial f/\partial x)^2/2.$$

Total energy density and power

Combining the results above gives the total energy density

$$\text{energy density} = \frac{1}{2}\mu\left[\left(\frac{\partial f}{\partial t}\right)^2 + \frac{F_T}{\mu}\left(\frac{\partial f}{\partial x}\right)^2\right] = \frac{1}{2}\mu\left[\left(\frac{\partial f}{\partial t}\right)^2 + v^2\left(\frac{\partial f}{\partial x}\right)^2\right]$$

The last step uses the fact (which we have not proved here) that $v^2 = F_T/\mu$.

Power is energy per unit time. The amount of energy that passes a given point in a unit time is the energy per unit length times the velocity

power=energy/time=(energy/length)(length)/time
=(energy/length)(length/time)=(energy density)(velocity).

$$\text{power} = \frac{1}{2} v\mu\left[\left(\frac{\partial f}{\partial t}\right)^2 + v^2\left(\frac{\partial f}{\partial x}\right)^2\right]$$

Actually, not that's quite right. The expression above is correct only if the wave has just a part moving to the right. If it has parts moving to the left and right, we need to make a better argument. There are two ways to get to the more general expression. In one method, one can integrate the expression for the energy density from x_1 to x_2 to get the energy in the interval $[x_1,x_2]$. Then the time derivative of the energy will be the power at x_1 minus the power at x_2. Taking the time derivative inside the integral, using the wave equation (see the wav equation document), and a partial integration leads to the correct answer. A less formal and more physical argument starts by observing that the power is force times velocity. The upward force of one bit of string on a bit to its right is $\text{upward force} = -F_T \frac{\partial f}{\partial x}$ while the upward velocity is $\frac{\partial f}{\partial t}$. Thus $\text{power} = -F_T \frac{\partial f}{\partial x}\frac{\partial f}{\partial t} = -\mu v^2 \frac{\partial f}{\partial x}\frac{\partial f}{\partial t}$. This is the general expression, but for a wave moving to the right, it is equal to the one above.

Example

For an example of what this means, consider a sine wave
$f(x,t) = A \sin(kx-\omega t)$ with $v = \omega/k$.

If you put this into the forms above for the energy density or for the power, you will find that they are proportional to A^2f^2, the square of the amplitude times the square of the frequency. That's not too hard to understand. Roughly speaking, a string element travels a distance 4A in a time 1/f. So the velocity is proportional to Af and the kinetic energy to the square of that. Similar (but slightly harder) reasoning leads to the conclusion that the potential energy is also proportional to A^2f^2.
(Note: If you use these results to compute the energy stored in one wave length of the sine wave, you will find the result that it is the same as what I got in class by more intuitive means except thet the "16" there is replaced by $(2\pi)^2/2=19.7$. So the argument was not so bad afterall even for the overall constant.)