FOURIER SERIES

Waves of any shape are possible. Harmonic waves are just a particular case. However, they are an especially interesting case because any wave can be represented as a sum of harmonic waves. This representation is called the *Fourier series*.
The basic idea is already familiar to you from your study of vectors. You know that any vector can be represented as a sum of basis vectors. In three dimensions, these are sometimes called $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Then we can write $\mathbf{v} = \mathbf{i}v_1 + \mathbf{j}v_2 + \mathbf{k}v_3$. The basis vectors are *orthonormal;* for example, $\mathbf{i} \cdot \mathbf{j} = 0$ and $\mathbf{i} \cdot \mathbf{i} = 1$. We will generalize this idea in two steps---first to vector spaces of arbitrary dimension and then to function spaces, which are vector spaces of infinite dimension.

Consider a vector space of dimension d. The unit vectors will now be called $\mathbf{e}_i$ with $i = 1,\ldots,d$. They are orthonormal so that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. The *Kronicker delta* on the right hand side is defined to be one if i and j are the same and zero otherwise. Now an arbitrary vector can be written $\mathbf{v} = \sum_{i=1}^{d} \mathbf{e}_i v_i$. We say that the basis vectors are *complete* when any vector can be written as such a sum. Conversely, we can get the components of $\mathbf{v}$ via $v_i = \mathbf{e}_i \cdot \mathbf{v}$. These are the two properties that we need of a basis---orthonormality and completeness.

The second, and much bigger step, is to generalize this idea from a space of vectors to a space of functions. The case we are working on is the waves on a string of finite length L with the boundary condition that the string is clamped at the ends. So the function space is all the possible shapes of the string---all functions $f(x)$ on [0,L] with $f(0)=f(L)=0$. One such function is the analogue of a vector in the previous discussion. We would like to have a basis of functions so that any such function f can be represented as a sum over the basis functions with some coefficients. As it happens, a very nice orthonormal and complete basis is the harmonic, standing waves that we have already discussed! Let’s see how that works.

Recall that the spatial part of the standing waves is given by the functions $\sin(n\pi x/L)$. These are essentially the basis functions. However, to save writing and to make these orthonormal, it is convenient to define the basis functions via

$$S_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}.$$

Direct integration gives the result

$$\int_0^L dx S_m(x) S_n(x) = \delta_{mn} \ .$$

This is the generalization to functions of the notion of orthonormal for vectors. If *m* and *n* are not the same, the integral is zero, and we say that the functions are orthogonal. If *m* and *n* are the same, then the integral is one, and we say that the functions are normalized to one. Since *n* can be any positive integer, there are an infinite number of basis functions. We say that the vector space of functions is infinite dimensional.

Just as the basis vectors were complete, so too are the basis functions complete. That means that any function $f(x)$ that satisfies the boundary conditions $f(0)=f(L)=0$ can be written as a sum over the basis functions with coefficients f_n.

$$f(x) = \sum_{n=1}^{\infty} f_n S_n(x) = \sum_{n=1}^{\infty} f_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right).$$

In the last form, $c_n = f_n\sqrt{\frac{2}{L}}$ is introduced for convenience. The proof of this is a little too much to put here. If you would like to discuss it, please come see me. At this point, it's not clear that we have done anything useful since the coefficients have not been given. However, with $f(x)$ given, the coefficients can be obtained by an integral with the basis functions, which is the analogue of finding the components of a vector by computing its dot product with a basis vector.

$$f_n = \int_0^L dx S_n(x) f(x) = \sqrt{\frac{2}{L}} \int_0^L dx \sin\left(\frac{n\pi x}{L}\right) f(x)$$

or

$$c_n = \frac{2}{L}\int_0^L dx \sin\left(\frac{n\pi x}{L}\right) f(x).$$

Thus, if we know $f(x)$, then we can find the c_n from this inversion formula and then use them to give the representation of f as a Fourier series. This can be very useful because the harmonic basis functions are simple and have very nice properties.

Now with this new tool, let's return to the physics problem at hand. The problem is to describe all the possible wave forms on the string of length L with the ends clamped. The standing wave solutions are of the form $\cos(\omega_n t)\sin\left(\frac{n\pi x}{L}\right)$. The angular frequency is $\omega_n = v\left(\frac{n\pi}{L}\right)$. You can easily check that another solution is $\sin(\omega_n t)\sin\left(\frac{n\pi x}{L}\right)$. The superposition principle tells us that any linear combination of these

$$D(x,t) = \sum_n \left[a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right] \sin\left(\frac{n\pi x}{L}\right) = \sum_n c_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

[where $c_n(t) \equiv a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$] is also a solution. Thus the function *D(x,t)* is a possible motion of the string, and at each t, this wave is a fourier series. The fourier coefficients $c_n(t)$ change with time as indicated. Since the basis functions are complete, any wave can be represented in this form! This includes all kinds of complicated solutions that don't look anything like standing waves. Nevertheless, we now know that they are linear combinations of standing waves.

This is great except for the problem that we now have a huge number of solutions to the wave equation. The problem manifests itself in the constants a_n, b_n . Any values for those give a valid solution D. The problem is resolved by using additional information. The constants are determined by the initial conditions, which are given in the two functions that give the shape of the string and the velocity of each piece of the string at t=0: $D(x,0)$ and $\frac{\partial}{\partial t} D(x,t)\Big|_{t=0}$. Setting t=0 in the form above or differentiating with respect to t and than setting t=0 gives

$$D(x,0) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \frac{\partial D(x,0)}{\partial t} = \sum_n \omega_n b_n \sin\left(\frac{n\pi x}{L}\right).$$

Now the a_n and the b_n can be determined by using the inversion formula given above

$$a_n = \frac{2}{L}\int_0^L dx \sin\left(\frac{n\pi x}{L}\right) D(x,0) \quad \text{and} \quad b_n = \frac{1}{\omega_n}\frac{2}{L}\int_0^L dx \sin\left(\frac{n\pi x}{L}\right)\frac{\partial D(x,0)}{\partial t}.$$

Thus, if you know the initial conditions, you can do these integrals to get the a_n and b_n coefficients. Then they are put back into the general expression for D(x,t) to give the evolution of the disturbance with time.

To close, I would like to emphasize that this method of fourier series is very general. It can be used to advantage in many other problems. You will certainly encounter it many more times in your studies.

FOURIER TRANSFORM

A closely related method is the fourier transform a.k.a. the fourier integral. It is the appropriate representation when the length of the interval L becomes infinite, i.e. we return to the case of waves on an infinite string. In that limit, the wave numbers $k_n = \frac{n\pi}{L}$ become very closely spaced. The sum on *n* becomes an integral on k . The representation of *D(x,t)* is as an integral over all the harmonic waves. This can be written in terms of trig functions, but it is more convenient and compact to use the complex exponential notation, which comes from the identity $e^{i\theta} = \cos\theta + i\sin\theta$. (If you do not recall this, or otherwise want more info on complex numbers, see sections 11.6 and 11.7 of your calculus text by Stein and Barcellos, 5th ed.)

$$D(x,t) = \int dk\left[a(k)e^{i\omega t} + b(k)e^{-i\omega t}\right]e^{ikx} \qquad \text{with} \qquad \omega = \omega(k) = vk\,.$$

This looks a lot different from the trig functions. You can verify that it is equivalent by using the identity to write out the exponentials. I do not recommend this. It is much easier to just stick this form into the wave equation and verify that it is a solution. As with the fourier series, this representation has wide application in mathematics, physics, other quantitative sciences, and engineering.