Blackbody radiation and the temperature of the universe

The question here is "What is the energy spectrum of a photon gas at temperature T?" More precisely, by energy spectrum is meant the energy per unit volume per unit frequency. It is called u here and is a function of the temperature and the angular frequency ω. Thus $u(T, \omega)d\omega dV$ is the energy of radiation in the volume dV and in the frequency interval from ω to $\omega + d\omega$. The radiation must be viewed as a photon gas and not as classical electromagnetic radiation in order to get the correct answer.

The calculation is a straightforward application of the canonical distribution to the photon gas. Most of the effort is in the technical issue of enumerating the photon states.

Consider a single photon mode with frequency ν and angular frequency $\omega = 2\pi\nu$. To describe it quantum mechanically, it is treated as a harmonic oscillator. We saw already that the energy levels of a quantum oscillator are nE. (n is a non-negative integer, and E is a basic unit of energy associated with the oscillator.) Planck told us that the photons associated with an EM wave of frequency ν have energy $E = h\nu = \hbar\omega$ each. h is Planck's constant, and $\hbar \equiv h/(2\pi) = 1.05 \times 10^{-34} Js$. Thus, if this mode is populated by n photons, the energy is $nE = n\hbar\omega$.

If the system is in contact with a heat reservoir with temperature T ($\beta \equiv 1/(kT)$, the probability of occupation n is given by the canonical distribution

$$P_n = e^{-\beta nE}/Z \qquad Z = \sum_n e^{-\beta nE}. \tag{1}$$

The average occupation is

$$\bar{n} = \sum_n nP_n = -\frac{1}{\beta}\frac{\partial}{\partial E}\ln\sum_n e^{-\beta nE} = -\frac{1}{\beta}\frac{\partial}{\partial E}\ln\frac{1}{1-e^{-\beta E}} = \frac{1}{e^{\beta E}-1} \tag{2}$$

(Verify these results by filling in the missing steps.)

The hard part of this problem is to determine the number of photon states per unit volume in the angular frequency interval from ω to $\omega + d\omega$. Let us put that problem aside for a minute and pretend that we know the answer and that it is given by the function $\mathcal{N}(\omega)d\omega$.

Thus, the number of photons per unit volume in the frequency interval $d\omega$ is the average number per mode times the number of modes

$$\rho d\omega \equiv \bar{n}(\hbar\omega)\mathcal{N}(\omega)d\omega. \tag{3}$$

The energy per unit volume and in the interval $d\omega$ around ω is the energy unit $E = \hbar\omega$ times the previous expression for the number density

$$u d\omega = \bar{n}(\hbar\omega)\hbar\omega\mathcal{N}(\omega)d\omega. \tag{4}$$

Now if we just knew $\mathcal{N}$, we would be done.

So let's try to determine $\mathcal{N}$. (This calculation is not really important, so it is ok to just skim it and use the result.) A mode with angular frequency ω has a related wavelength λ. It is more convenient to deal with the wavenumber defined by $k \equiv 2\pi/\lambda$. (Do not confuse this k with the Boltzmann constant that has already appeared above.) Then the angular frequency and the wave number are related by the speed of light $\omega = ck$. (This is equivalent to $\nu\lambda = c$.) The momentum associated with the mode is $p = \hbar k$, and then $E = cp$ as we saw already for the relation between energy and momentum for massless particles (photons) in relativity. On a line of length L, the standing waves have wave number $k = m\pi/L$ with $m = 1, 2, \ldots$. In a box with sides L, the wavenumber is a vector, and it's $\vec{k} = (\pi/L)(m_x, m_y, m_z)$. Thus, the density of states in $\vec{k}$ space is one per volume $(\pi/L)^3$ in $\vec{k}$-space. In the first octant shell between k and $k + dk$, there are

$$2\frac{1}{(\pi/L)^3}\frac{1}{8}(4\pi k^2 dk) \tag{5}$$

states. The 2 is for two polarizations. The electric field is transverse to the direction of propagation. There are two independent directions perpendicular to the direction of the wave propagation in which the electric field can point. The second factor is the density of states. The third factor accounts for the fact that all the m's are positive so that only the first octant is desired. The last factor is the volume of a spherical shell of radius k and thickness dk. Using $k = \omega/c$, in the expression above gives

$$\mathcal{N}(\omega)d\omega = \frac{8\pi}{(2\pi c)^3}\omega^2 d\omega \tag{6}$$

states per unit volume in the angular frequency interval $d\omega$.

Now we put this expression for $\mathcal{N}$ and the earlier result for $\bar{n}$ into the expression above for u to get that the energy per unit volume and in the interval $d\omega$ around ω is

$$u d\omega = \frac{1}{e^{\beta\hbar\omega}-1}\frac{8\pi\hbar\omega^3 d\omega}{(2\pi c)^3}. \tag{7}$$

This is the black body spectrum—a very important result. u is a function of T and ω. It goes to zero for $\omega \to 0$ and for $\omega \to \infty$. There is a peak around $\hbar\omega = 3kT$ Draw a picture to get a feel for the formula.

The total energy density is

$$u_0(T) = \int_0^\infty u(T,\omega)d\omega = \frac{\pi^2}{15}\frac{(kT)^4}{(c\hbar)^3} = \frac{4}{c}\sigma T^4. \tag{8}$$

$\sigma \equiv (\pi^2/60)(k^4/c^2)(1/\hbar)^3$ is the Stefan–Boltzmann constant.

Temperature of the universe

Note that expanding the box by a factor of x ($L' = xL$, $k' = k/x$, $\omega' = \omega/x$, and $d\omega' = d\omega/x$) has the same effect as lowering the temperature by a factor of x ($T \to T/x$). This follows from the condition that the number of photons in the box remain constant.

$$\begin{aligned}
L'^3\rho'(\omega')d\omega' &= L^3\rho(\omega)d\omega & (9)\\
L'^3\rho'(\omega')d\omega' &= (L'^3/x^3)\rho(x\omega')xd\omega' & (10)\\
\rho'(\omega') &= \rho(x\omega')/x^2 & (11)\\
\hbar\omega'\rho'(\omega',T) &= \hbar\omega'\rho(x\omega',T)/x^2 & (12)\\
&= \hbar\omega'\rho(\omega',T/x). & (13)
\end{aligned}$$

The last step uses the explicit form of ρ given above. Thus, when the universe expands by a factor of x, the temperature of the radiation falls to T/x. Shortly after the big bang, the temperature was very high. Since then, the universe has expanded and cooled. Presently, the temperature of the universe is about 2.7K. This is obtained by direct observation of the cosmic microwave background radiation and comparison with the blackbody spectrum to obtain T. Check that the peak of the spectrum for 2.7K really is in the microwave region.

Entropy of the photon gas

In the volume $V = L^3$, the energy is $U = u_0(T)V$. The entropy density $s(u_0)$ is a function of the energy density and

$$S(U,V) = Vs(u_0) = Vs(U/V) \tag{14}$$

Since

$$\frac{1}{T} = \left.\frac{\partial S}{\partial U}\right|_V = V \left.\frac{\partial s}{\partial U}\right|_V = V\frac{1}{V}\frac{\partial s}{\partial u_0}, \tag{15}$$

and since $T^4 = cu_0/(4\sigma)$, we have

$$\frac{\partial s}{\partial u_0} = [cu_0/(4\sigma)]^{-1/4}. \tag{16}$$

This integrates to

$$s = [c/(4\sigma)]^{-1/4}\frac{4}{3}u_0^{3/4} = [c/(4\sigma)]^{-1/4}\frac{4}{3}(4\sigma/c)^{3/4}T^3. \tag{17}$$

Thus,

$$s = \frac{16\sigma}{3c}T^3 \qquad S = \frac{16\sigma}{3c}T^3V. \tag{18}$$

Pressure of the photon gas

First we have to get S expressed in terms of U and V:

$$S = sV = (4\sigma/c)^{1/4}\frac{4}{3}(U/V)^{3/4}V = \frac{4}{3}(4\sigma/c)^{1/4}U^{3/4}V^{1/4} \tag{19}$$

Then the pressure is obtained from

$$P/T = \left.\frac{\partial S}{\partial V}\right|_U = \frac{4}{3}(4\sigma/c)^{1/4}U^{3/4}\frac{1}{4}V^{-3/4} = \frac{1}{3}(4\sigma/c)^{1/4}u_0^{3/4} \tag{20}$$

giving

$$P = \frac{1}{3}(\frac{4\sigma T^4}{c})^{1/4}u_0^{3/4} \tag{21}$$

or

$$P = \frac{1}{3}u_0 \tag{22}$$

Very nice!