Four-vector notation

First a three-dimensional notation: The unit vectors in the three spatial directions are often called $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. It is more common in physics to name them after the corresponding coordinates $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. I usually use the later notation.

In four-dimensional spacetime, it is very convenient to introduce a new notation. The familiar spacetime coordinates t, x, y, and z are given new names

$$x^0 = t \qquad x^1 = x \qquad x^2 = y \qquad x^3 = z \quad .$$

The upper numbers 0, 1, 2, and 3 are superscripts not powers. For an unspecified value of the superscript, Greek letters like μ and ν are used. They can take the values 0, 1, 2, or 3. Thus x^μ stands for any one of the spacetime coordinates.

This allows many expressions to be written in a more compact way. For example, consider the infinitesimal spacetime interval

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \ .$$

To rewrite this, we first introduce the *metric* $g_{\mu\nu}$. It has two subscripts, each of which can be 0, 1, 2, or 3. So there are sixteen elements. It is defined by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

This means that $g_{11} = 1$, $g_{12} = 0$, etc.
It allows us to write the infinitesimal spacetime interval as

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu}\, dx^\mu\, dx^\nu = g_{\mu\nu}\, dx^\mu\, dx^\nu \ .$$

In the last version, the Einstein summation convention is used. In it, an index repeated once up and once down is understood to be summed from 0 to 3.

At this point, it does not seem like much has really been done. No matter how efficient it may be, it’s still just notation at this point. However, as soon as you want to describe a space with curvature, you must use a metric. And the metric will have a form more complicated that that above in order to account for the curvature. And then, when you want to describe gravity, you must again use a metric, because in general relativity, the relativistic theory of gravity, gravity *is* curvature!