Lorentz transformations in four-vector notation

Recall that we have introduced the convenient notation with

$x^0 = t \qquad x^1 = x \qquad x^2 = y \qquad x^3 = z$

and with

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This means that $g_{11} = 1$, $g_{12} = 0$, etc.
It allows us to write the infinitesimal spacetime interval as

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu}\, dx^\mu\, dx^\nu = g_{\mu\nu}\, dx^\mu\, dx^\nu \ .$$

In the last version, the Einstein summation convention is used. In it, an index repeated once up and once down is understood to be summed from 0 to 3.

Since the Lorentz transformation is a linear transformation, it can be written in the form

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu \ .$$

The sum on ν is understood. This is four equations; one for each value of μ .There is also the same form for the transformation of the infinitesimals with x replaced by dx. To get this to reproduce the Lorentz transformation, we need to make the correct choice for Λ . You can check that the following form works:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since everyone agrees on the spacetime interval,

$$g_{\mu\nu}\, dx^{\mu} dx^{\nu} = g_{\alpha\beta} dx'^{\alpha} dx'^{\beta} = g_{\alpha\beta} \Lambda^{\alpha}{}_{\mu}\, dx^{\mu} \Lambda^{\beta}{}_{\nu}\, dx^{\nu} \quad .$$

Comparing the first and last forms and using the fact that the dx's can be anything, we conclude that

$$g_{\mu\nu} = g_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} \quad .$$

(Remember we are still using the summation convention!) To describe this equation in words, we say that the Minkowski metric is invariant under Lorentz transformations. You can check this for yourself by using the explicit forms of g and Λ.

Now this can be generalized. Any Λ that leaves g invariant is a Lorentz transformation. This frees us from the special choice of axes that we have been using. The relative velocity can be in any direction, and the two sets of spatial axes can be rotated relative to one another. Of course, in that case, the form of Λ is much more complicated than what we have used for our special coordinate convention. However, in most cases the four-vector notation helps us to do large parts of calculations without having to use the explicit and often messy forms for Λ.