Energy-Momentum in 4-vector notation

The main points here are the definition of the 4-vector of momentum (sometimes called energy-momentum) and its transformation properties using 4-vector notation. Just as the 3-vector notation that you have already learned in studying nonrelativistic physics is a lot easier to deal with than constantly writing out all the components, so also the 4-vector notation saves work and is cleaner. But it is equivalent to writing out all the components separately. You can get by with the version of things done in the text, but some calculations are easier using the more powerful notation.

First recall some of the stuff from the discussion of Lorentz transformations in 4-vector notation. (You might want to review that document before you read the rest of this.) We have introduced the convenient notation with

$x^0 = t \qquad x^1 = x \qquad x^2 = y \qquad x^3 = z$

and defined the Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Also remember the Einstein summation convention. In it, an index repeated once up and once down is understood to be summed from 0 to 3.

The Lorentz transformation of the 4-vector of spacetime position can also be written

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} .$$

The sum on ν is understood. This is four equations; one for each value of μ .There is also the same form for the transformation of the infinitesimals with x replaced by dx. To get this to reproduce the Lorentz transformation, we need to make the correct choice for Λ. You can check that the following form works:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If the situation at hand does not conform to the conventions of aligned axes and motion along the x-axis, then the form of Λ is more complicated.

The 4-vector of momentum has (not surprisingly) 4 components labeled by the index μ which runs from 0 to 3. It is defined by

$$P^{\mu} = m\frac{dx^{\mu}}{d\tau} \quad .$$

This is four expressions. While this definition has some nice properties, for most purposes, it is more convenient to start from the form

$$P^0 = \gamma m \quad P^1 = \gamma m v_x \quad P^2 = \gamma m v_y \quad P^3 = \gamma m v_z$$

which our text shows is the equivalent to the first form. In these relations, $\gamma = 1\big/\sqrt{1-v^2}$ with v the speed of the particle. It is also shown there that it is correct to identify the time component of the 4-momentum with energy so that $P^0 = E$. The square of the 4-vector of momentum is analogous to the spacetime interval

$$P^2 = g_{\mu\nu}P^{\mu}P^{\nu} = \left(P^0\right)^2 - \left(P^1\right)^2 - \left(P^2\right)^2 - \left(P^3\right)^2 = E^2 - \left|\vec{P}\right|^2 \quad .$$

(The summation convention is used twice, and only four of the sixteen terms are nonzero.) If P is the 4-momentum of a particle of mass m, then

$$P^2 = m^2 .$$

Now consider how the 4-momentum of a particle looks to two different observers with different inertial frames. The home frame sees the 4-vector P and the other frame sees P'. Our text shows that the 4-momentum transforms just like the spacetime position 4-vector. Since we already know that the expressions above for the transformation of x are the same as the component equations for the Lorentz transformation, all we have to do is to replace x by P to get

$$P'^{\mu} = \Lambda^{\mu}{}_{\nu}P^{\nu} \quad .$$

If you use the form of Λ given above and the summation convention and write out these four expressions, you will find that they are the same as the expressions in the book for the Lorentz transformation of the momentum 4-vector.

So why do all of this? The answer is the same as that to the question of why invent the notations algebra or calculus or 3-vectors: it makes calculations easier. However, for the purpose of this class, you will be able to get by without the 4-vector notation. Be warned though that without it, you will not be able to follow major parts of the next two lectures about general relativity and cosmology.