Variational principles and physics

We have seen that the constant velocity path between two spacetime events is, of all the paths that connect those two events, the one with the longest proper time. Since a free particle also moves at constant velocity, a free particle follows the path of longest proper time. This is an example of a variational principle in physics: of all the paths it could take, the actual path of the particle is the one with longest proper time. Suppose that we knew just that and did not already know that the path of longest proper time is the one with constant velocity, how might we find it? The method to use is called the calculus of variations. It turns out that all the laws of physics, including ones that you already know like Newton's laws, can be stated as variational principles.

The basic idea of the calculus of variations is the same as finding the place where a function has its maximum value. How is that done? Suppose the maximum of f(x) comes at $x=x_M$, then write $x=x_M+\delta$ and

$$f(x) = f(x_M + \delta) = f(x_M) + \frac{df(x_M)}{dx}\delta + \frac{1}{2}\frac{d^2 f(x_M)}{dx^2}\delta^2 + \ldots$$

We want the place x_M where there is no first order term in δ, i.e. where the first derivative is zero. Similarly, consider all the paths x(t) that connect two events $(t_0,x(t_0)=x_0)$ and $(t_1,x(t_1)=x_1)$ .Call the path with the longest proper time $x_M(t)$. Consider a nearby path $x(t)=x_M(t)+\delta(t)$ with δ(t) small and $\delta(t_0)=0$ and $\delta(t_1)=0$ so that x(t) connects the same two events that $x_M(t)$ does. The expression for the proper time for the path is

$$\Delta\tau = \int_{t_0}^{t_1} dt\sqrt{1-v(t)^2} \quad \text{with} \quad v(t) = \dot{x}_M(t) + \dot{\delta}(t)$$

The question is: For what path $x_M(t)$ is the proper time the largest. Just as for the calculus example, it will be the path for which the first order variation in δ(t) vanishes. So what you do is substitute the expression for v(t) into the expression for Δτ, expand out in δ, and demand that the coefficient of the term linear in δ is zero. The result is that $v_M(t)=dx_M(t)/dt$ must be a constant! So it gives back the answer that we knew it must.

Mechanics

Non-relativistic mechanics, Newton's laws, of motion can be formulated as a variational principle. Let T be the kinetic energy $T=mv^2/2$ and V(x) be the potential energy. Then define the *lagrangian* L=T-V and the *action* S

$$S = \int_{t_0}^{t_1} dt\, L = \int_{t_0}^{t_1} dt\, (T - V).$$

The variational principle for mechanics says that the trajectory of a particle is the one that minimizes the action. If you follow through the variational calculus to find out what that implies, you end up deriving Newton's second law F=ma!.

Optics

The rules of geometrical optics tell what happens when light rays encounter a boundary between two regions with different indices of refraction. The index of refraction n of a material is just a way to say what the speed of light is in the material $c_M=c/n$. You have probably already encountered these rules: Snells law and the law of reflection. There are lens formulas and other things too. Is there a general principle at work? Yes, and one way to state it is as a variational principle called *Fermat's principle*. This says that a ray of light, in traveling between two points, follows the path that minimizes the elapsed time.

Electricity and magnetism

You will study the electric and magnetic fields **E** and **B** in 9HD. The famous equations that give the dynamics of the electromagnetic fields are called Maxwell's equations. The are differential equations for **E** and **B**. Electrodynamics also has a variational formulation. The fields evolve in time in such a way that the action is minimized, and for the EM field, the action is

$$S = \int dt\, L = \int dt \int d^3x \left(\vec{E}^2 - \vec{B}^2\right)$$

Everything else

There are other more comprehensive theories, e.g. the Standard Model of elementary particle physics, for describing physics. The starting point for them is to give the lagrangian and say that the dynamics is determined by a variational principle that the action obtained as the integral of the lagrangian is a minimum. It is all very efficient and elegant.