Harmonic oscillator

There are two methods that are often used to do the harmonic oscillator. This is one of them. It uses the raising and lowering operators a_+ and a_-. Once it's all set up, many calculations can be done algebraically rather than with differential equations. It's a very nice method and generalizes well to other problems. This is not intended as a standalone document. Rather it is a supplement to the class discussion that has the equations recorded.

Observe that

$$-i\hbar\frac{\partial}{\partial x}\langle x|p\rangle = -i\hbar\frac{\partial}{\partial x}e^{ipx/\hbar} = pe^{ipx/\hbar} = p\langle x|p\rangle .$$

Define the operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$.

Then $\hat{p}f(x) = -i\hbar f'(x)$, where the prime means differentiation of f with respect to its argument. Now a little calculation gives

$\hat{p}(xf) = -i\hbar(f + xf')$ and $x\hat{p}f = -i\hbar xf'$

Thus $x\hat{p}f - \hat{p}(xf) = i\hbar f$

Since this is true for any function f(x), it can be written more abstractly as an algebraic relation between the operator x (just multiplication by x) and the operator $\hat{p}$.

$x\hat{p} - \hat{p}x = i\hbar$

This is a key relation in quantum mechanics. It is equivalent to the uncertainty relation. If the operator $\hat{p}$ were replaced by just multiplication by p as it is in classical mechanics, then $xp - px = 0$. The difference between two operators written in opposite order is called a commutator. And gets a special notation because it appears so often. For our case, the commutation relation above is written $[x,\hat{p}] = i\hbar$ with $[x,\hat{p}] \equiv x\hat{p} - \hat{p}x$. It's one of the most famous and useful relations in quantum mechanics. However, for this document, I'll refrain from using that notation. There's enough new stuff to deal with without that.

For the kinetic energy term in that Hamiltonian, we have

$$KE = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} = \frac{\hat{p}^2}{2m}$$

And for the full harmonic oscillator Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2x^2 = \frac{1}{2m}\left(\hat{p} + m^2\omega^2x^2\right)$$

Now we come to the important step, the definition of the raising and lowering operators.

$$a_+ \equiv \frac{1}{\sqrt{2m}}\left(\hat{p} + im\omega x\right) \quad a_- \equiv \frac{1}{\sqrt{2m}}\left(\hat{p} - im\omega x\right)$$

These have two great properties from which other neat results follow.

$$a_-a_+ - a_+a_- = \hbar\omega \quad \text{and} \quad H = a_+a_- + \frac{1}{2}\hbar\omega$$

Let's see how to get these relations.

$$a_+ a_- = \frac{1}{2m}\left(\hat{p} + im\omega x\right)\left(\hat{p} - im\omega x\right) = \frac{1}{2m}\left[\hat{p}^2 + im\omega\left(x\hat{p} - \hat{p}x\right) + m^2\omega^2 x^2\right]$$
$$= \frac{1}{2m}\left(\hat{p}^2 + m^2\omega^2 x^2\right) - \frac{1}{2}\hbar\omega = H - \frac{1}{2}\hbar\omega$$

Similarly

$$a_- a_+ = H + \frac{1}{2}\hbar\omega$$

Thus

$H = a_+ a_- + \frac{1}{2}\hbar\omega$ and $a_- a_+ - a_+ a_- = \hbar\omega$ as promised.

[Note: Sometimes the raising and lowering operators are normalized so that

$a_- a_+ - a_+ a_- = 1$ and $H = (a_+ a_- + \frac{1}{2})\hbar\omega$]

Now we will show the action of the raising and lowering operators and see where their names come from. Suppose that we have a state of definite energy and its wave function. $H\psi_E = E\psi_E$ Consider a new wave function $a_-\psi_E$. Let's get its energy.

$$H\left(a_-\psi_E\right) = \left(a_+ a_- + \frac{1}{2}\hbar\omega\right)\left(a_-\psi_E\right)$$

Several lines of messing around with the algebra of the raising and lowering operators gives $H\left(a_-\psi_E\right) = \left(E - \hbar\omega\right)\left(a_-\psi_E\right)$. This is very nice. It says that the new wave function is the wave function of a state with definite energy that is lower by $\hbar\omega$. I.e. $a_-\psi_E \propto \psi_{E-\hbar\omega}$. Similarly $a_+\psi_E \propto \psi_{E+\hbar\omega}$. Now you can see how the operators got their names. More importantly, we have shown that the energy levels of the harmonic oscillator are evenly spaced with separation $\hbar\omega$.

The next task is to find the energy and wave function for the ground state (state of lowest energy). Assuming there is a state of lowest energy, it must satisfy $a_-\psi_{E_0} = 0$. Otherwise there would be a state of lower energy. It follows that

$H\psi_{E_0} = \left(a_+ a_- + \frac{1}{2}\hbar\omega\right)\psi_{E_0} = \frac{1}{2}\hbar\omega\psi_{E_0}$ i.e. $E_0 = \frac{1}{2}\hbar\omega$. The lowest energy is not zero. This is very closely related to dark energy and to the cosmological constant problem. For many purposes, it's enough to know these results about the spectrum.

Now let's go for the wave function.

$$0 = a_- \psi = \frac{1}{\sqrt{2m}}(\hat{p} - im\omega x)\psi \text{ or}$$

$$-i\hbar \frac{\partial}{\partial x}\psi = im\omega x\psi$$

$$\hbar \frac{d\psi}{\psi} = -m\omega dx$$

$$\ln\psi = -\frac{m\omega}{2\hbar}x^2 + \text{constant}$$

$$\psi_{E_0} \propto e^{-\frac{m\omega}{2\hbar}x^2}$$

I do not know any algebraic way to get the normalization constant. You have to setup the integral

$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$$ rescale the integration variable, and look in your integral tables to get $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$

(Actually, if you know the trick, you can do that integral yourself. Hint: square it and change to plane polar coordinates.) The result is

$$\psi_{E_0} = Ce^{-\frac{m\omega}{2\hbar}x^2} \text{ with } C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

The final step is to get the proportionality constants in the actions of the raising and lowering operators. This is a bit involved. The result is important, but the details of the derivation are not important and not the least bit illuminating.

With shorthand $\psi_n(x) \equiv \psi_{E_n}(x)$

$$a_+\psi_n = \sqrt{(n+1)\hbar\omega}\,\psi_{n+1} \text{ and } a_-\psi_{n+1} = \sqrt{(n+1)\hbar\omega}\,\psi_n$$

The physically important thing is the factor $\sqrt{n+1}$. It's important for lasers, superconductors, and other macroscopic quantum systems.

Here's a way to do it. First some preliminary work:

$$\int dx\, \varphi^*(-im\omega x)\psi = \int dx\left[(im\omega x)\varphi\right]^*\psi$$

and

$$\int dx\,\varphi^*\hat{p}\psi = \int dx\,\varphi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\psi = \int dx\left(i\hbar\frac{\partial}{\partial x}\right)\varphi^*\psi + \varphi^*\psi\Big|_{\text{end points}} =$$

$$\int dx\left(-i\hbar\frac{\partial}{\partial x}\varphi\right)^*\psi = \int dx\,(\hat{p}\varphi)^*\psi$$

The surface term from the partial integration is zero because normalizable states vanish at the end points. When these to results are combined, we get

$$\int dx\left(a_+\varphi\right)^*\psi = \int dx\,\varphi^* a_-\psi$$

This is the result that we need.

We have already seen that $a_+\psi_n = A\psi_{n+1}$ with A the proportionality constant that we are trying to find.

$$A^*A = A^*A\int dx\,\psi^*_{n+1}\psi_{n+1} = \int dx\left(A\psi_{n+1}\right)^* A\psi_{n+1}$$

$$= \int dx\left(a_+\psi_n\right)^* a_+\psi_n = \int dx\,\psi^*_n a_- a_+\psi_n$$

$$= \int dx\,\psi^*_n\left(H + \frac{1}{2}\hbar\omega\right)\psi_n = \int dx\,\psi^*_n\left(\{n+1\}\hbar\omega\right)\psi_n$$

$$= (n+1)\hbar\omega$$

The phase of A does not matter. We can take A real so that $A^2 = (n+1)\hbar\omega$ and $A = \sqrt{(n+1)\hbar\omega}$ as claimed. The result for the lowering operator follows from an analogous calculation.

Now let's look some of the physics of the $\sqrt{n+1}$ factor. I'll use the laser example. The electromagnetic field can be treated as a collection of harmonic oscillators—one for each momentum $\vec{q}$ with wave number $\vec{k} = \vec{q}/\hbar$. I emphasize that this is the label for the oscillator. Please do not confuse it with the momentum p already discussed. It turns out that $\vec{q}$ is the actual momentum of a photon, while p has become more abstract and most definitely is not so directly physical. In what follows, do not think of the photon as some kind of particle feeling a harmonic oscillator force. In this description (a basic version of quantum field theory), the photon is a free quanton feeling no force at all. If all the harmonic oscillators for all the $\vec{q}$ values are in their n=0 ground states, there are no photons present. If the $\vec{q}$ oscillator is in its n=1 state, then there is a single photon with momentum $\vec{q}$. If the $\vec{q}$ and $\vec{q}'$ oscillators are in their n=1 levels, then there are two photons—one with momentum $\vec{q}$ and one with momentum $\vec{q}'$. And now for the important case: If the $\vec{q}$ oscillator is in the n=2 level, there are two photons present—each with momentum $\vec{q}$. You can generalize from here yourself.

For the oscillator with label $\vec{q}$, we take $\omega = c\left|\vec{k}\right| = c\left|\vec{q}\right|/\hbar$ and thus the energy steps are $\hbar\omega = c\left|\vec{q}\right|$ for the oscillator with label $\vec{q}$. So the angular frequency of the oscillator *does* match the angular frequency of its associated photon. Now finally think of the laser with a bunch of atoms in states with excitation energy $E = \hbar\omega$. When one of these atom emits a photon with energy E, it can become an excitation of any of the photon oscillators with momentum $\vec{q}$ such that $c\left|\vec{q}\right| = E$. Which one? Because of the $\sqrt{n+1}$, it is most likely to go into the $\vec{q}$ oscillator that already has the most excitation. Thus we tend to get a lot of photons with the same momentum $\vec{q}$, i.e. a coherent beam—a laser.