Quantum mechanics

Basic rules

In quantum mechanics we discuss processes. These are the kinds of things that might be done in experiments. A quanton or system of quantons is prepared in some way, and then a measurement is made on the system. Examples: prepare an electron with a definite momentum and then measure its position; prepare an atom with its spin along the z-axis, send it through a magnetic field and then measure its spin; prepare a proton and an anti proton with momenta p_1 and p_2 let them collide and then measure their momenta. All these have two parts: a preparation of one state with specified properties or values of some variables and then a measure of the state the system is in with respect to some other variables.

1. For each process, there is a probability amplitude a which is a complex number ($a=a_R+i\,a_I$). To spell out what the process is, we often write a in the following way

$$a = \langle \text{state } 2 | \text{state } 1 \rangle$$

and this is read “the amplitude that a system prepared in the state 1 will be observed to be in the state 2”. An example is $\langle x|p\rangle$, the amplitude that a particle with momentum p will be found at position x. The physical thing is the probability for the process

$$P = |\, a\,|^2 = a^* a = a_R^2 + a_I^2$$

Note that, as a probability should be, this is always a real, non-negative number.

2. If the *same* process can happen in many ways, the amplitudes for the ways add, *e.g.* if there are two ways 1 and 2 with amplitudes a_1 and a_2, then

$$a = a_1 + a_2 \qquad \text{and} \qquad P = |\, a_1 + a_2\,|^2$$

Note that P is not just the sum of $P_1=|a_1|^2$ and $P_2=|a_2|^2$ because there is a cross or interference term

$$P = a_1^* a_1 + a_2^* a_2 + a_1^* a_2 + a_2^* a_1 = P_1 + P_2 + 2\,\mathrm{Re}\,(a_1^* a_2)$$

All the interesting action is in the interference term. It is because of the interference term that probabilities do not combine in quantum mechanics as they do in classical probability, *e.g.* probability for the process in which the quanton can go through either slit is *not* the sum of the probabilities that it can go through each of the slits separately!

3. If a given way in which a process can occur is composed of several steps, then the amplitude for that way is the product of the amplitudes of each of the steps. For example, if a way has two steps α, with amplitude a_α followed by β with amplitude a_β, then the amplitude for that way is

$$a = a_\beta\, a_\alpha$$

These are the first basic rules, there are more to come, but these have in them already the essence of how quantum mechanics is different from classical physics.

They also hint at how it will be possible to do calculations in quantum mechanics. If we have amplitudes for some elementary processes, then we can find the amplitudes for much more complicated processes by combining them according to the rules 2 and 3.

What has this got to do with the wave function $\psi(x)$?

For each process, there is an amplitude. For example, there is the process "quanton prepared with momentum p and observed at position x". The amplitude is

$\langle$observed at x | prepared with p$\rangle$ or for short just $\langle x \mid p\rangle$.

(It turns out that this amplitude has the form

$$\langle x \mid p\rangle \propto e^{ipx/\hbar},$$

but that is not relevant right now.) Now generalize this to any way in which the state of the quanton is prepared. The notation for this amplitude is

$\langle$observed at x | prepared with in some state$\rangle$ = $\langle x \mid \psi\rangle = \psi(x)$.

This is the wave function. It's the amplitude to observe the quanton at x given some state preparation, which must be specified. Examples of the preparation are the one of definite p that we have already mentioned, or perhaps a definite energy level in an atom. There are lots of possibilities. The definite p wave function has the form

$$\langle x \mid p\rangle = \psi_p(x) \propto e^{ipx/\hbar}$$

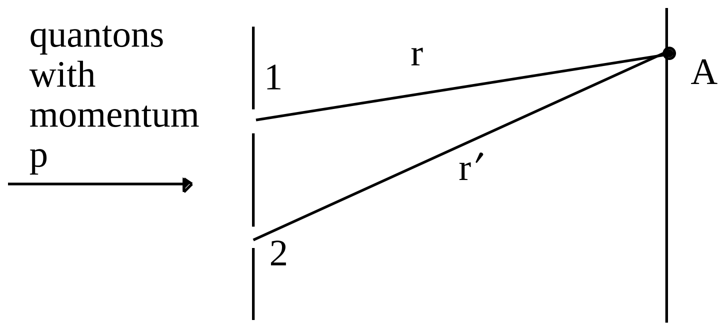
Thus the wave function is just a special case of the more general notion of an amplitude for a process.

Then the probability to observe the quanton in the interval of length dx at x is

$$P(x)\,dx = |\,\psi(x)\,|^2\,dx\,.$$

Notice that this is independent of x for the wave function with definite p. All x values are equally likely when the p value is precisely known. This is an extreme example of the uncertainty principle.

Two-slit problem



Using the basic rules, the amplitude for this process is

$$a = \langle A \mid 2\rangle \langle 2 \mid p\rangle + \langle A \mid 1\rangle \langle 1 \mid p\rangle .$$

Assuming we have set it up so that the amplitudes to arrive at 1 and 2 are equal

$$\langle 2 \mid p\rangle = \langle 1 \mid p\rangle$$

and using

$$\langle A \mid 1\rangle \propto e^{ipr/\hbar} \qquad \langle A \mid 2\rangle \propto e^{ipr'/\hbar} ,$$

we get

$$a \propto e^{ipr/\hbar} + e^{ipr'/\hbar}$$

and for the probability that the quanton is observed at A,

$$P = |\, a\,|^2 \propto (e^{-ipr/\hbar} + e^{-ipr'/\hbar})\,(e^{ipr/\hbar} + e^{ipr'/\hbar}) = 1 + 1 + 2\,\mathit{Re}\; e^{ip\Delta r/\hbar} = 2[1 + \cos(p\Delta r/\hbar)\,]$$

where $\Delta r = r' - r = d \sin\theta$, with d the slit separation and θ the observation angle as usual. Introducing the phase difference $\varphi = p\Delta r/\hbar = (2\pi\, d \sin\theta)/\lambda$, this becomes

$$P \propto \cos^2(\varphi/2)$$

just like we used to get for classical waves. Since some of the classical waves were light, which is made of the quantons that are photons, there better be such a relation!