Time dependence of amplitudes

Warm-up example

I will discuss a special apparatus called U. As we have seen, All the information about what U does is contained in the amplitudes iS | U | jS that tell what happens to S base states. There is equivalent information when other bases are used, but let's assume that somehow we know the situation in the S basis and that there is a very simple form

$$\langle iS|U|jS\rangle = \delta_{ij}\, e^{i\varphi_i}\ .$$

This says that U acts almost like an open S apparatus, except that all the channels or spin states are not treated the same. If a +S goes in, only a +S comes out, but the amplitude for that process is not 1, it is $e^{i\varphi_+}$. This still gives a probability of 1 but does have important effects in other experiments, as we will see in a moment.

Let's look at the following case.

$$\underset{T}{\begin{Bmatrix} +\| \\ 0 \\ -\| \end{Bmatrix}} \{U\} \underset{T'}{\begin{Bmatrix} +\| \\ 0 \\ -\| \end{Bmatrix}} = \underset{T}{\begin{Bmatrix} +\| \\ 0 \\ -\| \end{Bmatrix}} \underset{S}{\begin{Bmatrix} + \\ 0 \\ - \end{Bmatrix}} \{U\} \underset{S}{\begin{Bmatrix} + \\ 0 \\ - \end{Bmatrix}} \underset{T'}{\begin{Bmatrix} +\| \\ 0 \\ -\| \end{Bmatrix}}$$

In an equation, this is

$$\langle 0T'|U|0T\rangle = \sum_j \sum_i \langle 0T' \mid jS'\rangle\langle jS'|U|iS\rangle\langle iS|\,0T\rangle$$

Now this is actually progress. We have re-expressed the amplitude on the left, which we did not know, in terms of amplitudes that we do know. If for example, we continue with α=π/4 in the formulas (5.38) of the *Spin One* reading as we did in class, then we have values for the S,T amplitudes. For the case of relative rotation about the y-axis with the angle π/4, the jT | iS matrix is (i,j = +,0,-)

$$\begin{array}{cc} & \mathrm{i}\rightarrow \\ \mathrm{j}\downarrow & \begin{matrix} \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right) & \frac{1}{2} & \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right) \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right) & -\frac{1}{2} & \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right) \end{matrix} \end{array}$$

e.g. +T | 0S = 1/2

The S,U,S' amplitudes are above. Due to the δ, the nine terms collapse to three, and we get

$$\langle 0T'|U|0T\rangle = \sum_i \langle 0T'\,|\,iS'\rangle\langle iS'|U|iS\rangle\langle iS\,|\,0T\rangle = \sum_i \langle 0T'\,|\,iS'\rangle e^{i\varphi_i}\langle iS\,|\,0T\rangle$$

$$=\langle 0T'\,|+S'\rangle e^{i\varphi_+}\langle +S\,|\,0T\rangle + \langle 0T'\,|\,0S'\rangle e^{i\varphi_0}\langle 0S\,|\,0T\rangle + \langle 0T'\,|-S'\rangle e^{i\varphi_-}\langle -S\,|\,0T\rangle$$

$$=\left(-\frac{1}{2}\right)e^{i\varphi_+}\left(-\frac{1}{2}\right)+\frac{1}{\sqrt{2}}e^{i\varphi_0}\frac{1}{\sqrt{2}}+\frac{1}{2}e^{i\varphi_-}\frac{1}{2}=\frac{1}{2}\left[e^{i\varphi_0}+\frac{1}{2}\left(e^{i\varphi_+}+e^{i\varphi_-}\right)\right]$$

We see that in general, these phases really matter, and the probability is less than one. (There are very special choices of the phases for which the probability is 1.)

(There is a point that is a little tricky that should be mentioned. The amplitudes given above are of the type <jT|iS>. However, in these exppressions we need things of the form <jT'|iS'> and <iS|jT>. The amplitudes above are for the case where T is rotated by angle π/4 about the y-axis relative to S. Since that is the relation between T' and S', we can use those directly. For <iS|jT>, we must think a little more. We could observe that this T is also rotated by π/4 about y relative to this S and then use <iS|jT> = <jT|iS>* . Alternatively we could observe that the states on the left are rotated by -π/4 realtive to those on the right and use the (5.38) formulas with α → -α.. In fact, these two methods are equivalent.)

For an example of a nondiagonal amplitude, consider

$$\langle 0T'|U|+T\rangle = \sum_i \langle 0T'\,|\,iS'\rangle\langle iS'|U|iS\rangle\langle iS\,|+T\rangle = \sum_i \langle 0T'\,|\,iS'\rangle e^{i\varphi_i}\langle iS\,|+T\rangle$$

$$=\langle 0T'\,|+S'\rangle e^{i\varphi_+}\langle +S\,|+T\rangle + \langle 0T'\,|\,0S'\rangle e^{i\varphi_0}\langle 0S\,|+T\rangle + \langle 0T'\,|-S'\rangle e^{i\varphi_-}\langle -S\,|+T\rangle$$

$$=\left(-\frac{1}{2}\right)e^{i\varphi_+}\,\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}e^{i\varphi_0}\,\frac{1}{2}+\frac{1}{2}e^{i\varphi_-}\,\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)$$

$$=\frac{1}{2\sqrt{2}}\left[e^{i\varphi_0}+\frac{1}{\sqrt{2}}\left(-e^{i\varphi_+}\left(1+\frac{1}{\sqrt{2}}\right)+e^{i\varphi_-}\left(1-\frac{1}{\sqrt{2}}\right)\right)\right]$$

In general, this is not zero. If U acted like an open S, it would be zero.

What are the lessons here? First, this example shows that a process that multiplies the base states in one set by phases that are not all the same does have physical consequences. Second, it shows that even if the amplitudes for an apparatus are diagonal in one base set, they will not be diagonal in others. In this example, U is diagonal in the S base set (by assumption), but the calculation shows that it is not diagonal in the T base set.

Time dependence for states with definite energy

We have come to the crucial point. How do we begin to say something about the time dependence of amplitudes? We cannot derive the answer from something we already know because quantum mechanics is new and not contained in things we already know.

We must make an inspired guess and then test its consequences. If they agree with experiments, then we have discovered a new law of nature; if they do not, then we guess again.

We have one hint to get us going. Planck and Einstein showed that good results follow if we assume that electromagnetic waves are associated with quanta of energy E=hf. This is a relation between frequency (time dependence) and energy. Let's run with it. The time dependence of something with frequency f=ω/(2π) is cos(ωt) or sin(ωt) or $e^{i\omega t}$ or $e^{-i\omega t}$. Using the relation ω = 2π f = 2π E/h = E/ℏ, the last case is also written as

$$e^{-iEt/\hbar}.$$

Save that for a moment while we explore another point. Think about a quanton with a definite momentum and in particular momentum zero. We have already seen that the spatial dependence of such a state is $e^{ipx/\hbar} = 1$ for p=0. The quanton is equally likely to be found at any position. With p=0, the quanton is not moving, so it seems reasonable that the probability for it to be at any point should be independent of time. If the probability is independent of time, then the amplitude must be of the form $e^{i(something)}$. Further, a quanton with definite momentum also has definite energy. Together with the previous observation, we are motivated to try the hypothesis that the time dependence of an amplitude for a particle of definite energy should be

$$e^{-iEt/\hbar}.$$

Putting things together a little more, we can say that the amplitude to find a particle that has momentum p at position x at time t is

$$\langle x,t|p\rangle \quad e^{-i(Et-px)/\hbar}.$$

(Notice that the combination Et-px is a nice Lorentz scalar, so that it appears that form $e^{-iEt/\hbar}$ is more natural than the other possible choice $e^{iEt/\hbar}$.) The spatial dependence will concern us more later. Now we are more interested in the time dependence.

The conclusion of this discussion is that if a quanton has definite energy E, then the amplitude to find it in state ϕ at time t is

$$\langle \varphi,t|E\rangle \quad e^{-iEt/\hbar}.$$

Time evolution "apparatus"

We have talked in a general way about specific apparati like S and T and U, and a bit about how to handle a general apparatus A. Let's consider the special "apparatus", which is usually called the time evolution operator and which consists of just waiting for a time t. The notation is U(t). If we somehow know all the amplitudes j | U(t) | i for some set of base states, then we know everything about U(t). However, we have also emphasized that some base states may be much more convenient than others. Suppose that the base states all have definite energies E_i. Then if the quanton starts in state i, it will stay in state i because the amplitude is $e^{-iE_it/\hbar}$ and the probability to be in the state is independent of time. These are called *stationary states* because the probability to be in each of them is independent of time. Thus, in a basis of stationary states of definite energy the amplitudes for time evolution are

$$\langle j|U(t)|i\rangle = \delta_{ij} e^{-iE_i t/\hbar} \ .$$

A very large part of the industry of quantum mechanics is devoted to finding such stationary base states. That will concern us later. For now, we are trying to understand things in a more general way.

Notice that this structure is like the warm-up example that we did earlier. Each base state just gets a phase, but here the phase depends on time. Taking over the results from that discussion, we can make a couple statements. First, these phases have physical consequences. They don't do much in the base where the amplitudes are diagonal, but as soon as more complicated experiments are done, they have important physical consequences. Second, even if U(t) has diagonal amplitudes for some special base states, for other sets, that will no longer be true.

The typical situation that arises in quantum mechanics calculations is that there will be some set of base states where it is easy to say what U(t) is *i.e.* to give the amplitudes j | U(t) | i but that will *not* be the base set in which U(t) is diagonal. That sets up the problem already mentioned: Given the j | U(t) | i for some base set, find the base set of stationary states. The way that the problem is attacked is to consider very small changes in time and reduce the problem to a differential equation involving ∂ ϕt|ψ /∂t. It is called the Schroedinger equation.