

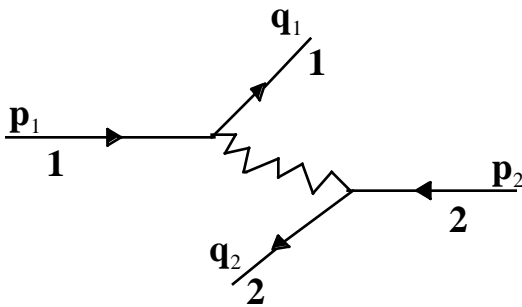
Identical particles, statistics, and spin

Introduction

When we consider processes involving *identical* particles, a new rule of quantum mechanics appears, and particles get divided into two classes: *bosons* and *fermions*. This leads to some very important effects. For bosons, there superfluidity, superconductivity, and lasers. For fermions, there is the *Pauli exclusion principle*, which has a profound effect on all of chemistry and on the stability of bulk matter.

Scattering of nonidentical particles

Consider the scattering of two distinguishable, spinless particles such as helium and oxygen nuclei off each other. A helium nucleus, which is two protons and two neutrons, is called an α -particle. The particles come in with momenta \mathbf{p}_1 and \mathbf{p}_2 and leave the collision with \mathbf{q}_1 and \mathbf{q}_2 . The amplitude for this process is $f(\mathbf{q}_1, \mathbf{q}_2; \mathbf{p}_1, \mathbf{p}_2)$.



The probability for this process is $P(\mathbf{q}_1, \mathbf{q}_2; \mathbf{p}_1, \mathbf{p}_2) = |f(\mathbf{q}_1, \mathbf{q}_2; \mathbf{p}_1, \mathbf{p}_2)|^2$. An alternative process would have particle 1 scattering to momentum \mathbf{q}_2 and particle 2 scattering to momentum \mathbf{q}_1 with amplitude $f(\mathbf{q}_2, \mathbf{q}_1; \mathbf{p}_1, \mathbf{p}_2)$ and with probability $P(\mathbf{q}_2, \mathbf{q}_1; \mathbf{p}_1, \mathbf{p}_2) = |f(\mathbf{q}_2, \mathbf{q}_1; \mathbf{p}_1, \mathbf{p}_2)|^2$.



Suppose that we do not care which particle ends up with momentum q_1 and we just want the probability to get some particle with q_1 and another particle with q_2 . Since particles 1 and 2 are not the same, these are distinguishable processes and the probabilities add

$$\begin{aligned} & P(\text{particle with } q_1, \text{ particle with } q_2; p_1, p_2) \\ &= P(q_1, q_2; p_1, p_2) + P(q_2, q_1; p_1, p_2) \\ &= |f(q_1, q_2; p_1, p_2)|^2 + |f(q_2, q_1; p_1, p_2)|^2. \end{aligned}$$

Scattering of identical particles and statistics

Now replace the oxygen nucleus with another helium nucleus. It is now the scattering of two identical α -particles. What is the probability for the process (which I will call process B),

$$\alpha(p_1) + \alpha(p_2) \rightarrow \alpha(q_1) + \alpha(q_2)$$

Since there is no way to tell the two α 's apart, there are two ways that this process can happen, and we must add the amplitudes to get the amplitude $\langle q_1, q_2 | p_1, p_2 \rangle$ for this α scattering $\langle q_1, q_2 | p_1, p_2 \rangle = f(q_1, q_2; p_1, p_2) + f(q_2, q_1; p_1, p_2)$. The probability is $P_B = |\langle q_1, q_2 | p_1, p_2 \rangle|^2 = |f(q_1, q_2; p_1, p_2) + f(q_2, q_1; p_1, p_2)|^2$.

Now notice that the amplitude

$$\langle q_1, q_2 | p_1, p_2 \rangle = f(q_1, q_2; p_1, p_2) + f(q_2, q_1; p_1, p_2)$$

is *symmetric* in the interchange of q_1 and q_2 :

$$\langle q_2, q_1 | p_1, p_2 \rangle = f(q_2, q_1; p_1, p_2) + f(q_1, q_2; p_1, p_2) = \langle q_1, q_2 | p_1, p_2 \rangle.$$

Identical particles, like α -particles, for which amplitudes have this symmetry are called *bosons*.

It is a truly amazing thing (which has no classical counterpart) that there is another possibility that appears in nature. If the experiment is done with protons rather than α -particles or other bosons, then the rule is to add the amplitudes for the two possible ways with a minus sign

$$\langle q_1, q_2 | p_1, p_2 \rangle = f(q_1, q_2; p_1, p_2) - f(q_2, q_1; p_1, p_2)$$

so that the amplitude is *antisymmetric* in the interchange of the momenta

$$\langle q_2, q_1 | p_1, p_2 \rangle = - \langle q_1, q_2 | p_1, p_2 \rangle!$$

Particles that follow this antisymmetry rule are called *fermions*.

We should think a little more carefully about this. Originally we had a rule that said that when there are two ways for the same process, then the amplitudes add. Now we know that was the rule when there was only one quanton involved. When there are identical quantons involved, the rule is generalized so that there is an extra minus sign when the two ways are related to each other by the exchange of a pair of identical fermions.

So all the particles of nature are divided into these two classes: bosons and fermions. For example, photons, gluons, W 's, Z 's, and gravitons are bosons, while protons, neutrons, electrons, muons, and quarks are fermions. This property of particles is

called their *statistics*. The statistics of a particle can be boson or fermion. By the way, the names come from two big-time physicists: Bose and Fermi. Also sometimes more names get into the act and Bose statistics is also called Bose-Einstein statistics, while Fermi becomes Fermi-Dirac. Einstein you know; Dirac is another physicist, who was one of the key people in the development of quantum mechanics.

Spin and statistics

Elementary particles have *spin*, which is the name that we give to the *intrinsic angular momentum* of the particle. It is the angular momentum that the particle has that does *not* come from its motion via the *orbital angular momentum* $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. The symbol used for spin is usually \mathbf{s} or \mathbf{S} . Like orbital angular momentum, it is a vector. Spin is quantized. Some particles, like the α -particle are spin zero. The electron is "spin-1/2", which means that the possible values of the projection of the spin angular momentum onto the z-axis are $S_z = \pm\hbar/2$. The W is spin-1 with three values $S_z = \pm\hbar$, and 0. The Δ is spin-3/2 with $S_z = \pm 3\hbar/2, \pm\hbar/2$.

Particles with highest $S_z = 1/2, 3/2, \dots$ are called half-integer spin particles, while those with highest $S_z = 0, 1, 2 \dots$ are called integer spin particles. This is another way in which all particles are divided into two classes: half-integer spin and integer spin.

Putting together these two spin classes with the statistics, it might appear that there are four types of particles. However, nature uses only two of these: all integer spin particles are bosons, and all half-integer spin particles are fermions. (Or at least so far, no particle with either of the other two combinations has been discovered.) There is also a theorem that gives the same result. It is called the *spin and statistics theorem*. There is no simple proof. It is proven within the structure of relativistic quantum field theory. The essential ingredients seem to be relativity and causality. Perhaps someday there will be a deeper understanding and a simpler proof.

Fermions and bosons with spin

Since we now know that there are no spinless fermions, we must revise our discussion of fermions to include the spin variable. The state of a spin-1/2 particle like a proton must specify the momentum \mathbf{p} and the spin projection S_z . The values for S_z are $\pm\hbar/2$ but sometimes people use just $\pm 1/2$ or even \uparrow and \downarrow to indicate the two values of S_z . To save typing here, I will use s for S_z . Also if we are going to discuss bosons with nonzero spin, the spin variable is needed.

Now the more general rules for fermions and bosons are that amplitudes involving identical bosons must be symmetric under the interchange of the variable pairs $(\mathbf{p}_1, s_1) \leftrightarrow (\mathbf{p}_2, s_2)$ while amplitudes for identical fermions must be antisymmetric.

In dealing with these matters, it is important to keep in mind that spin is an observable quantity so that process in which an initial or final state differ by only the value of one of the spin variables are distinguishable processes.

These considerations may seem somewhat abstract at this point, but the consequences are profound. For example, consider again the scattering of two protons. Suppose that the interaction of the protons cannot change the spin. (This is approximately correct at very low energy and, in a special sense, also at very high energy.). Suppose that the initial state is prepared with "spin up" *i.e.* $s=\uparrow$ for the two protons. Then in the final state, both the spins will still be up. With the assumption that the interaction does not depend on spin, then the functions f are independent of the spin variables. Thus

$$\langle (\mathbf{q}_1, \uparrow), (\mathbf{q}_2, \uparrow) | (\mathbf{p}_1, \uparrow), (\mathbf{p}_2, \uparrow) \rangle = f(\mathbf{q}_1, \mathbf{q}_2; \mathbf{p}_1, \mathbf{p}_2) - f(\mathbf{q}_2, \mathbf{q}_1; \mathbf{p}_1, \mathbf{p}_2).$$

To simplify a bit, look at this process in the center of momentum frame where $\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{p}_1 + \mathbf{p}_2 = 0$. Further consider the specific final state where the $\mathbf{q}_1, \mathbf{q}_2$ line is perpendicular to the $\mathbf{p}_1, \mathbf{p}_2$ line. In that special case, the two f amplitudes above are actually equal because they describe the same process of scattering through an angle of $\pi/2$. (It is not true for other angles between the \mathbf{q} and \mathbf{p} lines.) Then for that direction of final state particles, the whole amplitude vanishes, and no particles come out at angle $\pi/2$!

On the other hand, if it were bosons scattering at $\pi/2$, the probability would be $4 |f|^2$. This should be compared with the result of $2 |f|^2$ that would obtain for nonidentical particles when you don't care which particle went in which of the two directions at angle $\pi/2$.

More consequences

Bosons

Suppose that an excited atom is going drop to a lower energy level and emit a photon into a mode g in which there are no photons. Call the amplitude for that process A . What is the amplitude if there is already a photon in the mode g ? Even with no photon-photon interaction, the amplitude is not A ; it is $A\sqrt{2}$. In general, if the change is from n to $n+1$ photons in mode g , the amplitude is $A\sqrt{(n+1)}$. So if a mode already has a lot of photons in it, then it is more likely for another to go in. The same thing holds for other bosons. Unfortunately, even though this is just a consequence of bose statistics, I have not been able to concoct a simple proof that uses only things that you already know. This is the quantum mechanical effect that leads to superfluidity, superconductivity, and lasers.

Fermions

Consider a process in which two fermions start in any combined initial state ψ end up in states (E_1, s_1) and (E_2, s_2) . You can think of the E 's as energies, but that is not necessary; they are just some description of the states. We have

$$\langle (E_2, s_2), (E_1, s_1) | \psi \rangle = - \langle (E_1, s_1), (E_2, s_2) | \psi \rangle$$

so that if $E_2 = E_1$ and $s_2 = s_1$, then the amplitude is minus itself and therefore zero. This tells us that we cannot have two fermions in exactly the same state (including spin). This is called the *Pauli exclusion principle*. It is absolutely crucial to an

understanding of atoms with more than one electron and thus to chemistry. Without it, the atoms of different elements would be much more similar than they are, and chemistry would not be nearly so rich in its phenomena.

Comments

With these rules for dealing with identical particles, you now know all the structure of nonrelativistic quantum mechanics. There is a lot of machinery that has been developed for applying the theory to different problems, but that does not involve any fundamentally new ideas.

However, this theory does not cover the quantum behavior of relativistic particles. You might think that you could just write down a relativistic version of what we have already and be done. Nature has had other ideas. A proper treatment of relativistic quantum mechanics requires the baroque edifice of relativistic quantum field theory. When particles are relativistic, they have enough energy to create other particles, so a theory that can accommodate a changing number of particles is needed. One also needs to incorporate relativity and causality. Relativistic quantum field theory can do this. However, at this point you have not yet seen much about fields much less quantum fields. In 9HD, you will learn a great deal about the electromagnetic field. At the end of that, it would be possible to say a few things about quantum fields. The basic idea was used in our discussion of blackbody radiation last quarter. We have seen that standing waves or plane waves can, through Fourier analysis, be used as the basis for a description of any wave. Further, each of the modes by itself has the same structure as the harmonic oscillator. Then in quantum field theory, each plane wave or standing wave is treated separately and given a quantum mechanical treatment that is the same as what we are doing for the harmonic oscillator. (It is worth noting in passing that you can do an awful lot of physics if the only problem you know is the harmonic oscillator. It's an all time favorite with physicists. It can be solved exactly and relatively simply, and many other systems can be roughly approximated as harmonic oscillators.)

A topic of much current research in high energy physics is *supersymmetry*. Ordinary symmetries such as translating or rotating particles in space do not change the statistics of the particles. In supersymmetry, fermions and bosons are interchanged. Every particle has a friend, which is the same in all characteristics like charge and mass, but with the opposite statistics and with spin differing by a half unit of \hbar . The particle and its friend are interchanged in the supersymmetry transformation. It is not known if or how nature takes advantage of this possible symmetry. But we do know that it cannot be in a simple way because the particles we know, such as electrons, do not have such friends of the same mass. If the friends exist at all, they must be too massive to have been found with present accelerators, *i.e.* more than about 50 times the proton mass.